Algorithmic problems for free-abelian times free groups

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Abstract

We study direct products of free-abelian and free groups with special emphasis on algorithmic problems. After giving natural extensions of standard notions into that family, we find an explicit expression for an arbitrary endomorphism of $\mathbb{Z}^m \times F_n$. These tools are used to solve several algorithmic and decision problems for $\mathbb{Z}^m \times F_n$: the membership problem, the isomorphism problem, the finite index problem, the subgroup and coset intersection problems, the fixed point problem, and the Whitehead problem.

Contents

Introduction

Free-abelian groups, namely $\mathbb{Z}^m$, are classical and very well known. Free groups, namely F_n, are much wilder and have a much more complicated structure, but they have also been extensively studied in the literature since more than a hundred years ago. The goal of this paper is to investigate direct products of the form $\mathbb{Z}^m \times F_n$, namely free-abelian times free groups. At a first look, it may seem that many questions and problems concerning $\mathbb{Z}^m \times F_n$ will easily reduce to the corresponding questions or problems for $\mathbb{Z}^m$ and F_n; and, in fact, this is the case when the problem considered is easy or rigid enough. However, some other naive looking questions have a considerably more elaborated answer in $\mathbb{Z}^m \times F_n$ rather than in $\mathbb{Z}^m$ or F_n. This is the case, for example, when one considers automorphisms: $\mathrm{Aut}(\mathbb{Z}^m \times F_n)$ naturally contains $GL_m(\mathbb{Z}) \times \mathrm{Aut}(F_n)$, but there are many more automorphisms other than those preserving the factors $\mathbb{Z}^m$ and F_n. This fact causes potential complications when studying problems involving automorphisms: apart from understanding the problem in both the free-abelian and the free parts, one has to be able to control how is it affected by the interaction between the two parts.

Another example of this phenomena is the study of intersections of subgroups. It is well known that every subgroup of $\mathbb{Z}^m$ is finitely generated. This is not true for free groups F_n with $n \geqslant 2$, but it is also a classical result that all these groups satisfy the Howson property: the intersection of two finitely generated subgroups is again finitely generated. This elementary property fails dramatically in $\mathbb{Z}^m \times F_n$, when $m \geqslant 1$ and $n \geqslant 2$ (a very easy example reproduced below, already appears in [7] attributed to Moldavanski). Consequently, the algorithmic problem of computing intersections of finitely generated subgroups of $\mathbb{Z}^m \times F_n$ (including the preliminary decision problem on whether such intersection is finitely generated or not) becomes considerably more involved than the corresponding problems in $\mathbb{Z}^m$ (just consisting on a system of linear equations over the integers) or in F_n (solved by using the pull-back technique for graphs). This is one of the algorithmic problems addressed below (see Section 4).

Along all the paper we shall use the following notation and conventions. For $n \geqslant 1$, $[n]$ denotes the set integers $\{1, \ldots, n\}$. Vectors from $\mathbb{Z}^m$ will always be understood as row vectors, and matrices $\mathbf{M}$ will always be though as linear maps acting on the right, $\mathbf{v} \mapsto \mathbf{vM}$; accordingly, morphisms will always act on the right of the arguments, $x \mapsto x\alpha$. For notational coherence, we shall use uppercase boldface letters for matrices, and lowercase boldface letters for vectors (moreover, if $w \in F_n$ then $\mathbf{w} \in \mathbb{Z}^n$ will typically denote its abelianization). We shall use lowercase Greek letters for endomorphisms of free groups, $\phi\colon F_n \to F_n$, and uppercase Greek letters for endomorphisms of free-abelian times free groups, $\Phi\colon \mathbb{Z}^m \times F_n \to \mathbb{Z}^m \times F_n$.

The paper is organized as follows. In Section 1, we introduce the family of groups we are interested in, and we import there several basic notions and properties shared by both families of free-abelian, and free groups, such as the concepts of rank and basis, as well as the closeness property by taking

subgroups. In Section 2 we remind the folklore solution to the three classical Dehn problems within our family of groups. In the next two sections we study some other more interesting algorithmic problems: the finite index subgroup problem in Section 3, and the subgroup and the coset intersection problems in Section 4. In Section 5 we give an explicit description of all automorphisms, monomorphisms and endomorphisms of free-abelian times free groups which we then use in Section 6 to study the fixed subgroup of an endomorphism, and in Section 7 to solve the Whitehead problem within our family of groups.

1 Free-abelian times free groups

Let $T = \{ t_i \mid i \in I \}$ and $X = \{ x_j \mid j \in J \}$ be disjoint (possibly empty) sets of symbols, and consider the group G given by the presentation

$$G = \langle T, X \mid [T, T \sqcup X] \rangle,$$

where $[A, B]$ denotes the set of commutators of all elements from A with all elements from B. Calling Z and F the subgroups of G generated, respectively, by T and X, it is easy to see that Z is a free-abelian group with basis T, and F is a free group with basis X. We shall refer to the subgroups $Z = \langle T \rangle$ and $F = \langle X \rangle$ as the *free-abelian* and *free parts* of G, respectively. Now, it is straightforward to see that G is the direct product of its free-abelian and free parts, namely

$$G = \langle T, X \mid [T, T \sqcup X] \rangle \simeq Z \times F. \tag{1.1}$$

We say that a group is *free-abelian times free* if it is isomorphic to one of the form (1.1).

It is clear that in every word on the generators $T \sqcup X$, the letters from T can freely move, say to the left, and so every element from G decomposes as a product of an element from Z and an element from F, in a unique way. After choosing a well ordering of the set T (whose existence is equivalent to the axiom of choice), we have a natural *normal form* for the elements in G, which we shall write as $\mathbf{t}^{\mathbf{a}} w$, where $\mathbf{a} = (a_i)_i \in \bigoplus_{i \in I} \mathbb{Z}$, $\mathbf{t}^{\mathbf{a}}$ stands for the (finite) product $\Pi_{i \in I} t_i^{a_i}$ (in the given order for T), and w is a reduced free word on X.

Observe that the center of the group G is Z unless F is infinite cyclic, in which case G is abelian and so its center is the whole G. This exception will create some technical problems later on.

We shall mostly be interested in the finitely generated case, i.e. when T and X are both finite, say $I = [m]$ and $J = [n]$ respectively, with $m, n \geqslant 0$. In this case, Z is the free-abelian group of rank m, $Z = \mathbb{Z}^m$, F is the free group of rank n, $F = F_n$, and our group G becomes

$$G = \mathbb{Z}^m \times F_n = \langle t_1, \ldots, t_m, x_1, \ldots, x_n \mid t_i t_j = t_j t_i, \, t_i x_k = x_k t_i \rangle, \tag{1.2}$$

where $i, j \in [m]$ and $k \in [n]$. The normal form for an element $g \in G$ is now

$$g = \mathbf{t}^{\mathbf{a}} w = t_1^{a_1} \cdots t_m^{a_m} \, w(x_1, \ldots, x_n),$$

where $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ is a row integral vector, and $w = w(x_1, \ldots, x_n)$ is a reduced free word on the alphabet X. Note that the symbol $\mathbf{t}$ by itself has no real meaning, but it allows us to convert the notation for the abelian group $\mathbb{Z}^m$ from additive into multiplicative, by moving up the vectors (i.e. the entries of the vectors) to the level of exponents; this will be especially convenient when working in G, a noncommutative group in general.

Observe that the ranks of the free-abelian and free parts of G, namely m and n, are not invariants of the group G, since $\mathbb{Z}^m \times F_1 \simeq \mathbb{Z}^{m+1} \times F_0$. However, as one may expect, this is the only possible redundancy and so, we can generalize the concepts of rank and basis from the free-abelian and free contexts to the mixed free-abelian times free situation.

Observation 1.1. *Let Z and Z' be arbitrary free-abelian groups, and let F and F' be arbitrary free groups. If F and F' are not infinite cyclic, then*

$$Z \times F \simeq Z' \times F' \;\Leftrightarrow\; \mathrm{rk}(Z) = \mathrm{rk}(Z') \textit{ and } \mathrm{rk}(F) = \mathrm{rk}(F').$$

Proof. It is straightforward to see that the center of $Z \times F$ is Z (here is where $F \not\simeq \mathbb{Z}$ is needed). On the other hand, the quotient by the center gives $(Z \times F)/Z \simeq F$. The result follows immediately. □

Definition 1.2. Let $G = Z \times F$ be a free-abelian times free group and assume, without loss of generality, that $F \not\simeq \mathbb{Z}$. Then, according to the previous observation, the pair of cardinals (κ, ς), where κ is the abelian rank of Z and ς is the rank of F, is an invariant of G, which we shall refer to as the *rank* of G, $\mathrm{rk}(G)$. (We allow this abuse of notation because the rank of G in the usual sense, namely the minimal cardinal of a set of generators, is precisely $\kappa + \varsigma$: G is, in fact, generated by a set of $\kappa + \varsigma$ elements and, abelianizing, we get $G^{\mathrm{ab}} = (Z \times F)^{\mathrm{ab}} = Z \oplus F^{\mathrm{ab}}$, a free-abelian group of rank $\kappa + \varsigma$, so G cannot be generated by less than $\kappa + \varsigma$ elements.)

Definition 1.3. Let $G = Z \times F$ be a free-abelian times free group. A pair (A, B) of subsets of G is called a *basis* of G if the following three conditions are satisfied:

(i) A is an abelian basis of the center of G,

(ii) B is empty, or a free basis of a non-abelian free subgroup of G (note that this excludes the possibility $|B| = 1$),

(iii) $\langle A \cup B \rangle = G$.

In this case we shall also say that A and B are, respectively, the *free-abelian* and *free* components of (A, B). From (i), (ii) and (iii) it follows immediately

(iv) $\langle A \rangle \cap \langle B \rangle = \{1\}$,

(v) $A \cap B = \varnothing$,

since $\langle A \rangle \cap \langle B \rangle$ is contained in the center of G, but no non trivial element of $\langle B \rangle$ belongs to it.

Usually, we shall abuse notation and just say that $A \cup B$ *is a basis* of G. Note that no information is lost because we can retrieve A as the elements in $A \cup B$ which belong to the center of G, and B as the remaining elements.

Observe that, by (i), (iii) and (iv) in the previous definition, if (A, B) is a basis of a free-abelian times free group G, then $G = \langle A \rangle \times \langle B \rangle$; and by (i) and (ii), $\langle A \rangle$ is a free-abelian group and $\langle B \rangle$ is a free group not isomorphic to $\mathbb{Z}$; hence, by Observation 1.1, $\mathrm{rk}(G) = (|A|, |B|)$. In particular, this implies that $(|A|, |B|)$ does not depend on the particular basis (A, B) chosen.

On the other hand, the first obvious example is $T \cup X$ being a basis of the group $G = \langle T, X \mid [T, T \sqcup X] \rangle$ (note that if $|X| \neq 1$ then $A = T$ and $B = X$, but if $|X| = 1$ then $A = T \cup X$ and $B = \varnothing$ due to the technical requirement in Observation 1.1). We have proved the following.

Corollary 1.4. *Every free-abelian times free group G has bases and, every basis (A, B) of G satisfies* $\mathrm{rk}(G) = (|A|, |B|)$. □

Let us focus now our attention to subgroups. It is very well known that every subgroup of a free-abelian group is free-abelian; and every subgroup of a free group is again free. These two facts lead, with a straightforward argument, to the same property for free-abelian times free groups (this will be crucial for the rest of the paper).

Proposition 1.5. *The family of free-abelian times free groups is closed under taking subgroups.*

Proof. Let T and X be arbitrary disjoint sets, let G be the free-abelian times free group given by presentation (1.1), and let $H \leqslant G$.

If $|X| = 0, 1$ then G is free-abelian, and so H is again free-abelian (with rank less than or equal to that of G); the result follows.

Assume $|X| \geqslant 2$. Let $Z = \langle T \rangle$ and $F = \langle X \rangle$ be the free-abelian and free parts of G, respectively, and let us consider the natural short exact sequence associated to the direct product structure of G:

$$1 \longrightarrow Z \xrightarrow{\iota} Z \times F = G \xrightarrow{\pi} F \longrightarrow 1,$$

where ι is the inclusion, π is the projection $\mathbf{t}^{\mathbf{a}} w \mapsto w$, and therefore $\ker(\pi) = Z = \mathrm{Im}(\iota)$. Restricting this short exact sequence to $H \leqslant G$, we get

$$1 \longrightarrow \ker(\pi_{|H}) \xrightarrow{\iota} H \xrightarrow{\pi_{|H}} H\pi \longrightarrow 1,$$

where $1 \leqslant \ker(\pi_{|H}) = H \cap \ker(\pi) = H \cap Z \leqslant Z$, and $1 \leqslant H\pi \leqslant F$. Therefore, $\ker(\pi_{|H})$ is a free-abelian group, and $H\pi$ is a free group. Since $H\pi$ is free, $\pi_{|H}$ has a splitting

$$H \xleftarrow{\alpha} H\pi, \tag{1.3}$$

sending back each element of a chosen free basis for $H\pi$ to an arbitrary preimage.

Hence, α is injective, $H\pi\alpha \leqslant H$ is isomorphic to $H\pi$, and straightforward calculations show that the following map is an isomorphism:

$$\begin{array}{rcl} \Theta_\alpha \colon H & \longrightarrow & \ker(\pi_{|H}) \times H\pi\alpha \\ h & \longmapsto & \big(h(h\pi\alpha)^{-1},\, h\pi\alpha\big). \end{array} \tag{1.4}$$

Thus $H \simeq \ker(\pi_{|H}) \times H\pi\alpha$ is free-abelian times free and the result is proven. □

This proof shows a particular way of decomposing H into a direct product of a free-abelian subgroup and a free subgroup, which depends on the chosen splitting α, namely

$$H = (H \cap Z) \times H\pi\alpha. \tag{1.5}$$

We call the subgroups $H \cap Z$ and $H\pi\alpha$, respectively, the *free-abelian* and *free parts of H, with respect to the splitting α*. Note that the free-abelian and free parts of the subgroup $H = G$ with respect to the natural inclusion $G \hookleftarrow F \colon \alpha$ coincide with what we called the free-abelian and free parts of G.

Furthermore, Proposition 1.5 and the decomposition (1.5) give a characterization of the bases, rank, and all possible isomorphism classes of such an arbitrary subgroup H.

Corollary 1.6. *With the above notation, a subset $E \subseteq H \leqslant G = Z \times F$ is a basis of H if and only if*

$$E = E_Z \sqcup E_F,$$

where E_Z is an abelian basis of $H \cap Z$, and E_F is a free basis of $H\pi\alpha$, for a certain splitting α as in (1.3).

Proof. The implication to the left is straightforward, with $E = A \sqcup B$, and $(A, B) = (E_Z, E_F)$ except for the case $\operatorname{rk}(F) = 1$, when we have $(A, B) = (E_Z \sqcup E_F, \varnothing)$.

Suppose now that $E = A \sqcup B$ is a basis of H in the sense of Definition 1.3, and let us look at the decomposition (1.5), for suitable α. If $\operatorname{rk}(H\pi) = 1$, then H is abelian, A is an abelian basis for H, $B = \varnothing$ and all but exactly one of the elements in A belong to $H \cap Z$ (i.e. have normal forms using only letters from T); in this case the result is clear, taking E_F to be just that special element. Otherwise, $Z(H) = H \cap Z$ having A as an abelian basis; take $E_Z = A$ and $E_F = B$. It is clear that the projection $\pi \colon H \twoheadrightarrow H\pi$, $\mathbf{t}^{\mathbf{a}} u \mapsto u$, restricts to an isomorphism $\pi|_{\langle B \rangle} \colon \langle B \rangle \to H\pi$ since no nontrivial element in $\langle B \rangle$ belong to $\ker \pi = H \cap Z$. Therefore, taking $\alpha = \pi|_{\langle B \rangle}^{-1}$, E_F is a free basis of $H\pi\alpha$. □

Corollary 1.7. *Let G be the free-abelian times free group given by presentation* (1.1)*, and let $\operatorname{rk}(G) = (\kappa, \varsigma)$. Every subgroup $H \leqslant G$ is again free-abelian times free with $\operatorname{rk}(H) = (\kappa', \varsigma')$ where,*

(i) *in case of $\varsigma = 0$: $0 \leqslant \kappa' \leqslant \kappa$ and $\varsigma' = 0$;*

(ii) *in case of $\varsigma \geqslant 2$: either $0 \leqslant \kappa' \leqslant \kappa + 1$ and $\varsigma' = 0$, or $0 \leqslant \kappa' \leqslant \kappa$ and $0 \leqslant \varsigma' \leqslant \max\{\varsigma, \aleph_0\}$ and $\varsigma' \neq 1$.*

Furthermore, for every such (κ', ς'), there is a subgroup $H \leqslant G$ such that $\mathrm{rk}(H) = (\kappa', \varsigma')$. $\square$

Along the rest of the paper, we shall concentrate on the finitely generated case. From Proposition 1.5 we can easily deduce the following corollary, which will be useful later.

Corollary 1.8. *A subgroup H of $\mathbb{Z}^m \times F_n$ is finitely generated if and only if its projection to the free part $H\pi$ is finitely generated.* $\square$

The proof of Proposition 1.5, at least in the finitely generated case, is completely algorithmic; i.e. if H is given by a finite set of generators, one can effectively choose a splitting α, and compute a basis of the free-abelian and free parts of H (w.r.t. α). This will be crucial for the rest of the paper, and we make it more precise in the following proposition.

Proposition 1.9. *Let $G = \mathbb{Z}^m \times F_n$ be a finitely generated free-abelian times free group. There is an algorithm which, given a subgroup $H \leqslant G$ by a finite family of generators, it computes a basis for H and writes both, the new elements in terms of the old generators, and the old generators in terms of the new basis.*

Proof. If $n = |X| = 0, 1$ then G is free-abelian and the problem is a straightforward exercise in linear algebra. So, let us assume $n \geqslant 2$.

We are given a finite set of generators for H, say $\mathbf{t}^{\mathbf{c_1}} w_1, \ldots, \mathbf{t}^{\mathbf{c_p}} w_p$, where $p \geqslant 1$, $\mathbf{c_1}, \ldots, \mathbf{c_p} \in \mathbb{Z}^m$ are row vectors, and $w_1, \ldots, w_p \in F_n$ are reduced words on $X = \{x_1, \ldots, x_n\}$. Applying suitable Nielsen transformations, see [18], we can algorithmically transform the p-tuple $(w_1, \ldots, w_p)$ of elements from F_n, into another of the form $(u_1, \ldots, u_{n'}, 1, \ldots, 1)$, where $\{u_1, \ldots, u_{n'}\}$ is a free basis of $\langle w_1, \ldots, w_p \rangle = H\pi$, and $0 \leqslant n' \leqslant p$. Furthermore, reading along the Nielsen process performed, we can effectively compute expressions of the new elements as words on the old generators, say $u_j = \eta_j(w_1, \ldots, w_p)$, $j \in [n']$, as well as expressions of the old generators in terms of the new free basis, say $w_i = \nu_i(u_1, \ldots, u_{n'})$, for $i \in [p]$.

Now, the map $\alpha \colon H\pi \to H$, $u_j \mapsto \eta_j(\mathbf{t}^{\mathbf{c_1}} w_1, \ldots, \mathbf{t}^{\mathbf{c_p}} w_p)$ can serve as a splitting in the proof of Proposition 1.5, since $\eta_j(\mathbf{t}^{\mathbf{c_1}} w_1, \ldots, \mathbf{t}^{\mathbf{c_p}} w_p) = \mathbf{t}^{\mathbf{a_j}} \eta_j(w_1, \ldots, w_p) = \mathbf{t}^{\mathbf{a_j}} u_j \in H$, where $\mathbf{a_j}$, $j \in [n']$, are integral linear combinations of $\mathbf{c_1}, \ldots, \mathbf{c_p}$.

It only remains to compute an abelian basis for $\ker(\pi_{|H}) = H \cap \mathbb{Z}^m$. For each one of the given generators $h = \mathbf{t}^{\mathbf{c_i}} w_i$, compute $h(h\pi\alpha)^{-1} = \mathbf{t}^{\mathbf{d_i}}$ (here, we shall need the words ν_i computed before). Using the isomorphism Θ_α from the proof of Proposition 1.5, we deduce that $\{\mathbf{t}^{\mathbf{d_1}}, \ldots, \mathbf{t}^{\mathbf{d_p}}\}$ generate $H \cap \mathbb{Z}^m$; it only remains to use a standard linear algebra procedure, to extract from here an abelian basis $\{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}\}$ for $H \cap \mathbb{Z}^m$. Clearly, $0 \leqslant m' \leqslant m$.

We immediately get a basis (A, B) for H (with just a small technical caution): if $n' \neq 1$, take $A = \{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}\}$ and $B = \{\mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} u_{n'}\}$; and if $n' = 1$ take $A = \{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}, \mathbf{t}^{\mathbf{a_1}} u_1\}$ and $B = \varnothing$.

On the other hand, as a side product of the computations done, we have the expressions $\mathbf{t}^{\mathbf{a_j}} u_j = \eta_j(\mathbf{t}^{\mathbf{c_1}} w_1, \ldots, \mathbf{t}^{\mathbf{c_p}} w_p)$, $j \in [n']$. And we can also compute expressions of the $\mathbf{t}^{\mathbf{b_i}}$'s in terms of the $\mathbf{t}^{\mathbf{d_i}}$'s, and of the $\mathbf{t}^{\mathbf{d_i}}$'s in terms of the $\mathbf{t}^{\mathbf{c_i}} w_i$'s. Hence we can compute expressions for each one of the new elements in terms of the old generators.

For the other direction, we also have the expressions $w_i = \nu_i(u_1, \ldots, u_{n'})$, for $i \in [p]$. Hence, $\nu_i(\mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} u_{n'}) = \mathbf{t}^{\mathbf{e_i}} w_i$ for some $\mathbf{e_i} \in \mathbb{Z}^m$. But $H \ni (\mathbf{t}^{\mathbf{c_i}} w_i)(\mathbf{t}^{\mathbf{e_i}} w_i)^{-1} = \mathbf{t}^{\mathbf{c_i} - \mathbf{e_i}} \in \mathbb{Z}^m$, so we can compute integers $\lambda_1, \ldots, \lambda_{m'}$ such that $\mathbf{c_i} - \mathbf{e_i} = \lambda_1 \mathbf{b_1} + \cdots + \lambda_{m'} \mathbf{b_{m'}}$. Thus, $\mathbf{t}^{\mathbf{c_i}} w_i = \mathbf{t}^{\mathbf{c_i} - \mathbf{e_i}} \mathbf{t}^{\mathbf{e_i}} w_i = \mathbf{t}^{\lambda_1 \mathbf{b_1} + \cdots + \lambda_{m'} \mathbf{b_{m'}}} \mathbf{t}^{\mathbf{e_i}} w_i = (\mathbf{t}^{\mathbf{b_1}})^{\lambda_1} \cdots (\mathbf{t}^{\mathbf{b_{m'}}})^{\lambda_{m'}} \nu_i(\mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} u_{n'})$, for $i \in [p]$. ☐

As a first application of Proposition 1.9, free-abelian times free groups have solvable *membership problem*. Let us state it for an arbitrary group G.

Problem 1.10 (**Membership Problem,** $\mathrm{MP}(G)$)**.** *Given elements $g, h_1, \ldots, h_p \in G$, decide whether $g \in H = \langle h_1, \ldots, h_p \rangle$ and, in this case, computes an expression of g as a word on the h_i's.*

Proposition 1.11. *The Membership Problem for $G = \mathbb{Z}^m \times F_n$ is solvable.*

Proof. Write $g = \mathbf{t}^{\mathbf{a}} w$. We start by computing a basis for H following Proposition 1.9, say $\{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}, \mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} u_{n'}\}$. Now, check whether $g\pi = w \in H\pi = \langle u_1, \ldots, u_{n'} \rangle$ (membership is well known for free groups). If the answer is negative then $g \notin H$ and we are done. Otherwise, a standard algorithm for membership in free groups gives us the (unique) expression of w as a word on the u_j's, say $w = \omega(u_1, \ldots, u_{n'})$. Finally, compute $\omega(\mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} u_{n'}) = \mathbf{t}^{\mathbf{c}} w \in H$. It is clear that $\mathbf{t}^{\mathbf{a}} w \in H$ if and only if $\mathbf{t}^{\mathbf{a} - \mathbf{c}} = (\mathbf{t}^{\mathbf{a}} w)(\mathbf{t}^{\mathbf{c}} w)^{-1} \in H$ that is, if and only if $\mathbf{a} - \mathbf{c} \in \langle \mathbf{b_1}, \ldots, \mathbf{b_{m'}} \rangle \leqslant \mathbb{Z}^m$. This can be checked by just solving a system of linear equations; and, in the affirmative case, we can easily find an expression for g in terms of $\{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}, \mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} u_{n'}\}$, like at the end of the previous proof. Finally, it only remains to convert this into an expression of g in terms of $\{h_1, \ldots, h_p\}$ using the expressions we already have for the basis elements in terms of the h_i's. ☐

Corollary 1.12. *The membership problem for arbitrary free-abelian times free groups is solvable.*

Proof. We have $G = Z \times F$, where $Z = \langle T \rangle$ is an arbitrary free-abelian group and $F = \langle X \rangle$ is an arbitrary free group. Given elements $g, h_1, \ldots, h_p \in G$, let $\{t_1, \ldots, t_m\}$ (resp. $\{x_1, \ldots, x_n\}$) be the finite set of letters in T (resp. in X) used by them. Obviously all these elements, as well as the subgroup $H = \langle h_1, \ldots, h_p \rangle$, live inside $\langle t_1, \ldots, t_m \rangle \times \langle x_1, \ldots, x_n \rangle \simeq \mathbb{Z}^m \times F_n$ and we can restrict our attention to this finitely generated environment. Proposition 1.11 completes the proof. ☐

To conclude this section, let us introduce some notation that will be useful later. Let H be a finitely generated subgroup of $G = \mathbb{Z}^m \times F_n$, and consider a basis for H,

$$\{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}, \mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} u_{n'}\}, \tag{1.6}$$

where $0 \leqslant m' \leqslant m$, $\{\mathbf{b_1}, \ldots, \mathbf{b_{m'}}\}$ is an abelian basis of $H \cap \mathbb{Z}^m \leqslant \mathbb{Z}^m$, $0 \leqslant n'$, $\mathbf{a_1}, \ldots, \mathbf{a_{n'}} \in \mathbb{Z}^m$, and $\{u_1, \ldots, u_{n'}\}$ is a free basis of $H\pi \leqslant F_n$. Let $L = \langle \mathbf{b_1}, \ldots, \mathbf{b_{m'}} \rangle \leqslant \mathbb{Z}^m$ (with additive notation, i.e. these are true vectors with m integral coordinates each), and let us denote by $\mathbf{A}$ the $n' \times m$ integral matrix whose rows are the $\mathbf{a_i}$'s,

$$\mathbf{A} = \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{n'}} \end{pmatrix} \in \mathcal{M}_{n' \times m}(\mathbb{Z}).$$

If ω is a word on n' letters (i.e. an element of the abstract free group $F_{n'}$), we will denote by $\omega(u_1, \ldots, u_{n'})$ the element of $H\pi$ obtained by replacing the i-th letter in ω by u_i, $i \in [n']$. And we shall use boldface, $\boldsymbol{\omega}$, to denote the abstract abelianization of ω, which is an integral vector with n' coordinates, $\boldsymbol{\omega} \in \mathbb{Z}^{n'}$ (not to be confused with the image of $\omega(u_1, \ldots, u_{n'}) \in F_n$ under the abelianization map $F_n \twoheadrightarrow \mathbb{Z}^n$). Straightforward calculations provide the following result.

Lemma 1.13. *With the previous notations, we have*

$$H = \{\mathbf{t}^{\mathbf{a}}\, \omega(u_1, \ldots, u_{n'}) \mid \omega \in F_{n'}, \mathbf{a} \in \boldsymbol{\omega}\mathbf{A} + L\},$$

a convenient description of H. □

Definition 1.14. Given a subgroup $H \leqslant \mathbb{Z}^m \times F_n$, and an element $w \in F_n$, we define the *abelian completion of w in H* as

$$\mathcal{C}_{w,H} = \{\mathbf{a} \in \mathbb{Z}^m \mid t^{\mathbf{a}} w \in H\} \subseteq \mathbb{Z}^m.$$

Corollary 1.15. *With the above notation, for every $w \in F_n$ we have*

(i) *if $w \notin H\pi$, then $\mathcal{C}_{w,H} = \varnothing$,*

(ii) *if $w \in H\pi$, then $\mathcal{C}_{w,H} = \boldsymbol{\omega}\mathbf{A} + L$, where $\boldsymbol{\omega}$ is the abelianization of the word ω which expresses $w \in F_n$ in terms of the free basis $\{u_1, \ldots, u_{n'}\}$ (i.e. $w = \omega(u_1, \ldots, u_{n'})$; note the difference between w and ω).*

Hence, $\mathcal{C}_{w,H} \subseteq \mathbb{Z}^m$ is either empty or an affine variety with direction L (i.e. a coset of L). □

2 The three Dehn problems

We shall dedicate the following sections to solve several algorithmic problems in $G = \mathbb{Z}^m \times F_n$. The general scheme will be reducing the problem to the analogous

problem on each part, $\mathbb{Z}^m$ and F_n, and then apply the vast existing literature for free-abelian and free groups. In some cases, the solutions for the free-abelian and free parts will naturally build up a solution for G, while in some others the interaction between both will be more intricate and sophisticated; everything depends on how complicated becomes the relation between the free-abelian and free parts, with respect to the problem.

From the algorithmic point of view, the statement "let G be a group" is not sufficiently precise. The algorithmic behavior of G may depend on how it is given to us. For free-abelian times free groups, we will always assume that they are finitely generated and given to us with the standard presentation (1.2). We will also assume that the elements, subgroups, homomorphisms and other objects associated with the group are given to us in terms of this presentation.

As a first application of the existence and computability of bases for finitely generated subgroups of G, we already solved the membership problem (see Corollary 1.12), which includes the word problem. This last one, together with the conjugacy problem, are quite elementary because of the existence of algorithmically computable normal forms for the elements in G. The third of Dehn's problems is also easy within our family of groups.

Proposition 2.1. *Let $G = \mathbb{Z}^m \times F_n$. Then*

(i) *the word problem for G is solvable,*

(ii) *the conjugacy problem for G is solvable,*

(iii) *the isomorphism problem is solvable within the family of finitely generated free-abelian times free groups.*

Proof. As seen above, every element from G has a normal form, easily computable from an arbitrary expression in terms of the generators. Once in normal form, $\mathbf{t}^{\mathbf{a}}u$ equals 1 if and only if $\mathbf{a} = \mathbf{0}$ and u is the empty word. And $\mathbf{t}^{\mathbf{a}}u$ is conjugate to $\mathbf{t}^{\mathbf{b}}v$ if and only if $\mathbf{a} = \mathbf{b}$ and u and v are conjugate in F_n. This solves the word and conjugacy problems in G.

For the isomorphism problem, let $\langle X \mid R \rangle$ and $\langle Y \mid S \rangle$ be two arbitrary finite presentations of free-abelian times free groups G and G' (i.e. we are given two arbitrary finite presentations plus the information that both groups are free-abelian times free). So, both G and G' admit presentations of the form (1.2), say $\mathcal{P}_{n,m}$ and $\mathcal{P}_{n',m'}$, for some integers $m, n, m', n' \geqslant 0$, $n, n' \neq 1$ (unknown at the beginning). It is well known that two finite presentations present the same group if and only if they are connected by a finite sequence of Tietze transformations (see [18]); so, there exist finite sequences of Tietze transformations, one from $\langle X \mid R \rangle$ to $\mathcal{P}_{n,m}$, and another from $\langle Y \mid S \rangle$ to $\mathcal{P}_{n',m'}$ (again, unknown at the beginning). Let us start two diagonal procedures exploring, respectively, the tree of all possible Tietze transformations successively aplicable to $\langle X \mid R \rangle$ and $\langle Y \mid S \rangle$. Because of what was just said above, both procedures will necessarily reach presentations of the desired form in finite time. When knowing the

parameters m, n, m', n' we apply Observation 1.1 and conclude that $\langle X \mid R \rangle$ and $\langle Y \mid S \rangle$ are isomorphic if and only if $n = n'$ and $m = m'$. (This is a brute force algorithm, very far from being efficient from a computational point of view.) $\square$

3 Finite index subgroups

In this section, the goal is to find an algorithm solving the Finite Index Problem in a free-abelian times free group G:

Problem 3.1 (**Finite Index Problem,** $\mathrm{FIP}(G)$)**.** *Given a finite list $w_1, \ldots, w_s$ of elements in G, decide whether the subgroup $H = \langle w_1, \ldots, w_s \rangle$ is of finite index in G and, if so, compute the index an a system of right (or left) coset representatives for H.*

To start, we remind that this same algorithmic problem is well known to be solvable both for free-abelian and for free groups. Given several vectors $\mathbf{w_1}, \ldots, \mathbf{w_s} \in \mathbb{Z}^m$, the subgroup $H = \langle \mathbf{w_1}, \ldots, \mathbf{w_s} \rangle$ is of finite index in $\mathbb{Z}^m$ if and only if it has rank m. And here is an algorithm to make such a decision, and (in the affirmative case) to compute the index $[\mathbb{Z}^m : H]$ and a set of coset representatives for H: consider the $s \times m$ integral matrix $\mathbf{W}$ whose rows are the $\mathbf{w_i}$'s, and compute its Smith normal form, i.e. $\mathbf{PW} = \mathbf{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0)\mathbf{Q}$, where $\mathbf{P} \in \mathrm{GL}_s(\mathbb{Z})$, $\mathbf{Q} \in \mathrm{GL}_m(\mathbb{Z})$, $d_1, \ldots, d_r$ are non-zero integers each dividing the following one, $d_1 \mid d_2 \mid \cdots \mid d_r \neq 0$, the diagonal matrix has size $s \times m$, and $r = \mathrm{rk}(\mathbf{W}) \leqslant \min\{s, m\}$ (fast algorithms are well known to compute all these from $\mathbf{W}$, see [1] for details). Now, if $r < m$ then $[\mathbb{Z}^m : H] = \infty$ and we are done. Otherwise, H is the subgroup generated by the rows of ($\mathbf{W}$ and so those of) $\mathbf{PW}$, i.e. the image under the automorphism $Q\colon \mathbb{Z}^m \to \mathbb{Z}^m$, $\mathbf{v} \mapsto \mathbf{vQ}$ of the subgroup H' generated by the simple vectors $(d_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_m)$. It is clear that $[\mathbb{Z}^m : H'] = d_1 d_2 \cdots d_m$, with $\{(r_1, \ldots, r_m) \mid r_i \in [d_i]\}$ being a set of coset representatives for H'. Hence, $[\mathbb{Z}^m : H] = d_1 d_2 \cdots d_m$ as well, with $\{(r_1, \ldots, r_m)\mathbf{Q} \mid r_i \in [d_i]\}$ being a set of coset representatives for H.

On the other hand, the subgroup $H = \langle w_1, \ldots, w_s \rangle \leqslant F_n$ has finite index if and only if every vertex of the core of the Schreier graph for H, denoted $\mathcal{S}(H)$, are complete (i.e. have degree $2n$); this is algorithmically checkable by means of fast algorithms. And, in this case, the labels of paths in a chosen maximal tree T from the basepoint to each vertex (resp. from each vertex to the basepoint) give a set of left (resp. right) coset representatives for H, whose index in F_n is then the number of vertices of $\mathcal{S}(H)$. For details, see [24] for the classical reference or [16] for a more modern and combinatorial approach.

Hence, $\mathrm{FIP}(\mathbb{Z}^m)$ and $\mathrm{FIP}(F_n)$ are solvable. In order to build an algorithm to solve the same problem in $\mathbb{Z}^m \times F_n$, we shall need some well known basic facts about indices of subgroups that we state in the following two lemmas. For a subgroup $H \leqslant G$ of an arbitrary group G, we will write $H \leqslant_{\text{f.i.}} G$ to denote $[G : H] < \infty$.

Lemma 3.2. *Let G and G' be arbitrary groups, $\rho\colon G \twoheadrightarrow G'$ an epimorphism between them, and let $H \leqslant G$ and $H' \leqslant G'$ be arbitrary subgroups. Then,*

(i) $[G' : H\rho] \leqslant [G : H]$*; in particular, if $H \leqslant_{\text{f.i.}} G$ then $H\rho \leqslant_{\text{f.i.}} G'$.*

(ii) $[G' : H'] = [G : H'\rho^{-1}]$*; in particular, $H' \leqslant_{\text{f.i.}} G'$ if and only if $H'\rho^{-1} \leqslant_{\text{f.i.}} G$.* □

Lemma 3.3. *Let Z and F be arbitrary groups, and let $H \leqslant Z \times F$ be a subgroup of their direct product. Then*

$$[Z \times F : H] \leqslant [Z : H \cap Z] \cdot [F : H \cap F],$$

and

$$H \leqslant_{\text{f.i.}} Z \times F \;\Leftrightarrow\; H \cap Z \leqslant_{\text{f.i.}} Z \text{ and } H \cap F \leqslant_{\text{f.i.}} F.$$

Proof. It is straightforward to check that the map

$$\begin{array}{rcl} Z/(H \cap Z) \;\times\; F/(H \cap F) & \to & (Z \times F)/H \\ \big(z \cdot (H \cap Z)\,,\, f \cdot (H \cap F)\big) & \mapsto & zf \cdot H \end{array} \tag{3.1}$$

is well defined and onto; the inequality and one implication follow immediately. The other implication is a well know fact. □

Let $G = \mathbb{Z}^m \times F_n$, and let H be a subgroup of G. If $H \leqslant_{\text{f.i.}} G$ then, applying Lemma 3.2 (i) to the canonical projections $\tau\colon G \twoheadrightarrow \mathbb{Z}^m$ and $\pi\colon G \twoheadrightarrow F_n$, we have that both indices $[\mathbb{Z}^m : H\tau]$ and $[F_n : H\pi]$ must also be finite. Since we can effectively compute generators for $H\pi$ and for $H\tau$, and we can decide whether $H\tau \leqslant_{\text{f.i.}} \mathbb{Z}^m$ and $H\pi \leqslant_{\text{f.i.}} F_n$ hold, we have two effectively checkable necessary conditions for H to be of finite index in G: if either $[\mathbb{Z}^m : H\tau]$ or $[F_n : H\pi]$ is infinite, then so is $[G : H]$.

Nevertheless, these two necessary conditions together are not sufficient to ensure finiteness of $[G : H]$, as the following easy example shows: take $H = \langle sa, tb \rangle$, a subgroup of $G = \mathbb{Z}^2 \times F_2 = \langle s, t \mid [s,t] \rangle \times \langle a, b \mid \rangle$. It is clear that $H\tau = \mathbb{Z}^2$ and $H\pi = F_2$ (so, both indices are 1), but the index $[\mathbb{Z}^2 \times F_2 : H]$ is infinite because no power of a belongs to H.

Note that $H \cap \mathbb{Z}^m \leqslant H\tau \leqslant \mathbb{Z}^m$ and $H \cap F_n \leqslant H\pi \leqslant F_n$ and, according to Lemma 3.3, the conditions really necessary, and sufficient, for H to be of finite index in G are

$$H \leqslant_{\text{f.i.}} G \;\Leftrightarrow\; \begin{cases} H \cap \mathbb{Z}^m \leqslant_{\text{f.i.}} \mathbb{Z}^m, \\ H \cap F_n \leqslant_{\text{f.i.}} H\pi, \text{ and } H\pi \leqslant_{\text{f.i.}} F_n, \end{cases} \tag{3.2}$$

both stronger than $H\tau \leqslant_{\text{f.i.}} \mathbb{Z}^m$ and $H\pi \leqslant_{\text{f.i.}} F_n$ respectively (and none of them satisfied in the example above). This is the main observation which leads to the following result.

Theorem 3.4. *The Finite Index Problem for $\mathbb{Z}^m \times F_n$ is solvable.*

Proof. From the given generators for H, we start by computing a basis of H (see Proposition 1.9),

$$\{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}, \mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} u_{n'}\},$$

where $0 \leqslant m' \leqslant m$, $0 \leqslant n' \leqslant p$, $L = \langle \mathbf{b_1}, \ldots, \mathbf{b_{m'}} \rangle \simeq \mathbb{Z}^{m'}$ with abelian basis $\{\mathbf{b_1}, \ldots, \mathbf{b_{m'}}\}$, $\mathbf{a_1}, \ldots, \mathbf{a_{n'}} \in \mathbb{Z}^m$, and $H\pi = \langle u_1, \ldots, u_{n'} \rangle \simeq F_{n'}$ with free basis $\{u_1, \ldots, u_{n'}\}$. As above, let us write $\mathbf{A}$ for the $n' \times m$ integral matrix whose rows are $\mathbf{a_i} \in \mathbb{Z}^m$, $i \in [n']$.

Note that $L = \langle \mathbf{b_1}, \ldots, \mathbf{b_{m'}} \rangle \simeq H \cap \mathbb{Z}^m$ (with the natural isomorphism $\mathbf{b} \mapsto \mathbf{t}^{\mathbf{b}}$, changing the notation from additive to multiplicative). Hence, the first necessary condition in (3.2) is $\mathrm{rk}(L) = m$, i.e. $m' = m$. If this is not the case, then $[G : H] = \infty$ and we are done. So, let us assume $m' = m$ and compute a set of (right) coset representatives for L in $\mathbb{Z}^m$, say $\mathbb{Z}^m = \mathbf{c_1} L \sqcup \cdots \sqcup \mathbf{c_r} L$.

Next, check whether $H\pi = \langle u_1, \ldots, u_{n'} \rangle$ has finite index in F_n (by computing the core of the Schreier graph of $H\pi$, and checking whether is it complete or not). If this is not the case, then $[G : H] = \infty$ and we are done as well. So, let us assume $H\pi \leqslant_{\text{f.i.}} F_n$, and compute a set of right coset representatives for $H\pi$ in F_n, say $F_n = v_1(H\pi) \sqcup \cdots \sqcup v_s(H\pi)$.

According to (3.2), it only remains to check whether the inclusion $H \cap F_n \leqslant H\pi$ has finite or infinite index. Call $\rho\colon F_{n'} \twoheadrightarrow \mathbb{Z}^{n'}$ the abstract abelianization map for the free group of rank n' (with free basis $\{u_1, \ldots, u_{n'}\}$), and $A\colon \mathbb{Z}^{n'} \to \mathbb{Z}^m$ the linear mapping $\mathbf{v} \mapsto \mathbf{v}\mathbf{A}$ corresponding to right multiplication by the matrix $\mathbf{A}$. Note that

$$H \cap F_n = \{w \in F_n \mid \mathbf{0} \in \mathcal{C}_{w,H}\} = \{w \in F_n \mid \boldsymbol{\omega}\mathbf{A} \in L\} \leqslant H\pi,$$

where $\boldsymbol{\omega} = \omega\rho$ is the abelianization of the word ω which expresses w in the free basis $\{u_1, \ldots, u_{n'}\}$ of $H\pi$, i.e. $F_n \ni w = \omega(u_1, \ldots, u_{n'})$, see Corollary 1.15. Thus, $H \cap F_n$ is, in terms of the free basis $\{u_1, \ldots, u_{n'}\}$, the successive full preimage of L, first by the map A and then by the map ρ, namely $(L)A^{-1}\rho^{-1}$, see the following diagram:

$$\begin{array}{ccccccc} F_n \geqslant H\pi & \simeq & F_{n'} & \xrightarrow{\ \rho\ } \!\!\!\!\to & \mathbb{Z}^{n'} & \xrightarrow{\ A\ } & \mathbb{Z}^m \\ \triangledown\!\!/ & & \triangledown\!\!/ & & \triangledown\!\!/ & & \triangledown\!\!/ \\ H \cap F_n & \simeq & (L)A^{-1}\rho^{-1} & \longleftarrow\!\!| & (L)A^{-1} & \longleftarrow\!\!| & L \end{array} \tag{3.3}$$

Hence, using Corollary 3.2 (ii), $[H\pi : H \cap F_n] = [F_{n'} : (L)A^{-1}\rho^{-1}]$ is finite if and only if $[\mathbb{Z}^{n'} : (L)A^{-1}]$ is finite. And this happens if and only if $\mathrm{rk}((L)A^{-1}) = n'$. Since $\mathrm{rk}((L)A^{-1}) = \mathrm{rk}((L \cap \mathrm{Im}(A))A^{-1}) = \mathrm{rk}(L \cap \mathrm{Im}(A)) + \mathrm{rk}(\ker(A))$, we can immediately check whether this rank equals n', or not. If this is not the case, then $[H\pi : H \cap F_n] = [F_{n'} : (L)A^{-1}\rho^{-1}] = [\mathbb{Z}^{n'} : (L)A^{-1}] = \infty$ and we are done. Otherwise, $(L)A^{-1} \leqslant_{\text{f.i.}} \mathbb{Z}^{n'}$ and so, $H \cap F_n \leqslant_{\text{f.i.}} H\pi$ and $H \leqslant_{\text{f.i.}} G$.

Finally, suppose $H \leqslant_{\text{f.i.}} G$ and let us explain how to compute a set of right coset representatives for H in G (and so, the actual value of the index $[G : H]$). Having followed the algorithm described above, we have $\mathbb{Z}^m = \mathbf{c_1} L \sqcup \cdots \sqcup \mathbf{c_r} L$ and $F_n = v_1(H\pi) \sqcup \cdots \sqcup v_s(H\pi)$. Furthermore, from the situation in the previous paragraph, we can compute a set of (right) coset representatives for $(L)A^{-1}$ in $\mathbb{Z}^{n'}$, which can be easily converted (see Lemma 3.2 (ii)) into a set of right coset representatives for $H \cap F_n$ in $H\pi$, say $H\pi = w_1(H \cap F_n) \sqcup \cdots \sqcup w_t(H \cap F_n)$.

Hence, $F_n = \bigsqcup_{j \in [s]} \bigsqcup_{k \in [t]} v_j w_k (H \cap F_n)$, and $[F_n : H \cap F_n] = st$. Combining this with $\mathbb{Z}^m = \bigsqcup_{i \in [r]} \mathbf{t}^{\mathbf{c_i}} (H \cap \mathbb{Z}^m)$, and using the map in the proof of Lemma 3.3, we get $G = \mathbb{Z}^m \times F_n = \bigcup_{i \in [r]} \bigcup_{j \in [s]} \bigcup_{k \in [t]} \mathbf{t}^{\mathbf{c_i}} v_j w_k H$.

It only remains a cleaning process in the family of rst elements $\{\mathbf{t}^{\mathbf{c_i}} v_j w_k \mid i \in [r],\, j \in [s],\, k \in [t]\}$ to eliminate possible duplications as representatives of right cosets of H (this can be easily done by several applications of the membership problem for H, see Corollary 1.12). After this cleaning process, we get a genuine set of right coset representatives for H in G, and the actual value of $[G : H]$ (which is at most rst).

Finally, inverting all of them we will get a set of left coset representatives for H in G. □

Regarding the computation of the index $[G : H]$, we remark that the inequality among indices in Lemma 3.3 may be strict, i.e. $[G : H]$ may be strictly less than rst, as the following example shows.

Example 3.5. Let $G = \mathbb{Z}^2 \times F_2 = \langle s, t \mid [s,t] \rangle \times \langle a, b \mid \rangle$ and consider the (normal) subgroups $H = \langle s, t^2, a, b^2, bab \rangle$ and $H' = \langle s, t^2, a, b^2, bab, tb \rangle = \langle s, t^2, a, tb \rangle$ of G (with bases $\{s, t^2, a, b^2, bab\}$ and $\{s, t^2, a, tb\}$, respectively). We have $H \cap \mathbb{Z}^2 = H' \cap \mathbb{Z}^2 = \langle s, t^2 \rangle \leqslant_2 \mathbb{Z}^2$, and $H \cap F_2 = H' \cap F_2 = \langle a, b^2, bab \rangle \leqslant_2 F_2$, but

$$[\mathbb{Z}^2 \times F_2 : H] = 4 = [\mathbb{Z}^2 : H \cap \mathbb{Z}^2] \cdot [F_2 : H \cap F_2],$$

while

$$[\mathbb{Z}^2 \times F_2 : H'] = 2 < 4 = [\mathbb{Z}^2 : H' \cap \mathbb{Z}^2] \cdot [F_2 : H' \cap F_2],$$

with (right) coset representatives $\{1, b, t, tb\}$ and $\{1, t\}$, respectively. This shows that both the equality and the strict inequality can occur in Lemma 3.3.

4 The coset intersection problem and Howson's property

Consider the following two related algorithmic problems in an arbitrary group G:

Problem 4.1 (**Subgroup Intersection Problem,** $\mathrm{SIP}(G)$)**.** *Given finitely generated subgroups H and H' of G (by finite sets of generators), decide whether the intersection $H \cap H'$ is finitely generated and, if so, compute a set of generators for it.*

Problem 4.2 (**Coset Intersection Problem,** $\mathrm{CIP}(G)$)**.** *Given finitely generated subgroups H and H' of G (by finite sets of generators), and elements $g, g' \in G$, decide whether the right cosets gH and $g'H'$ intersect trivially or not; and in the negative case (i.e. when $gH \cap g'H' = g''(H \cap H')$), compute such a $g'' \in G$.*

A group G is said to have the *Howson property* if the intersection of every pair (and hence every finite family) of finitely generated subgroups $H, H' \leqslant_{\text{f.g.}} G$ is again finitely generated, $H \cap H' \leqslant_{\text{f.g.}} G$.

It is obvious that $\mathbb{Z}^m$ satisfies Howson property, since every subgroup is free-abelian of rank less than or equal to m (and so, finite). Moreover, $\mathrm{SIP}(\mathbb{Z}^m)$ and $\mathrm{CIP}(\mathbb{Z}^m)$ just reduce to solving standard systems of linear equations.

The case of free groups is more interesting. Howson himself established in 1954 that F_n also satisfies the Howson property, see [14]. Since then, there has been several improvements of this result in the literature, both about shortening the upper bounds for the rank of the intersection, and about simplifying the arguments used. The modern point of view is based on the pull-back technique for graphs: one can algorithmically represent subgroups of F_n by the core of their Schreier graphs, and the graph corresponding to $H \cap H'$ is the pull-back of the graphs corresponding to H and H', easily constructible from them. This not only confirms Howson's property for F_n (namely, the pull-back of finite graphs is finite) but, more importantly, it provides the algorithmic aspect into the topic by solving $\mathrm{SIP}(F_n)$. And, more generally, an easy variation of these arguments using pullbacks also solves $\mathrm{CIP}(F_n)$, see Proposition 6.1 in [5].

Baumslag [2] established, as a generalization of Howson's result, the conservation of Howson's property under free products, i.e. if G_1 and G_2 satisfy Howson property then so does $G_1 * G_2$. Despite it could seem against intuition, the same result fails dramatically when replacing the free product by a direct product. And one can find an extremely simple counterexample for this, in the family of free-abelian times free groups; the following observation is folklore (it appears in [7] attributed to Moldavanski, and as the solution to exercise 23.8(3) in [4]).

Observation 4.3. *The group $\mathbb{Z}^m \times F_n$, for $m \geqslant 1$ and $n \geqslant 2$, does not satisfy the Howson property.*

Proof. In $\mathbb{Z} \times F_2 = \langle t \mid \rangle \times \langle a, b \mid \rangle$, consider the (finitely generated) subgroups $H = \langle a, b \rangle$ and $H' = \langle ta, b \rangle$. Clearly,

$$\begin{aligned} H \cap H' &= \{w(a,b) \mid w \in F_2\} \cap \{w(ta,b) \mid w \in F_2\} \\ &= \{w(a,b) \mid w \in F_2\} \cap \{t^{|w|_a} w(a,b) \mid w \in F_2\} \\ &= \{t^0 w(a,b) \mid w \in F_2,\ |w|_a = 0\} \\ &= \langle\langle b \rangle\rangle_{F_2} = \langle a^{-k} b a^k,\ k \in \mathbb{Z} \rangle, \end{aligned}$$

where $|w|_a$ is the total a-exponent of w (i.e. the first coordinate of the abelianization $\mathbf{w} \in \mathbb{Z}^2$ of $w \in F_2$). It is well known that the normal closure of b in F_2 is

not finitely generated, hence $\mathbb{Z} \times F_2$ does not satisfy the Howson property. Since $\mathbb{Z} \times F_2$ embeds in $\mathbb{Z}^m \times F_n$ for all $m \geqslant 1$ and $n \geqslant 2$, the group $\mathbb{Z}^m \times F_n$ does not have this property either. $\square$

We remark that the subgroups H and H' in the previous counterexample are both isomorphic to F_2. So, interestingly, the above is a situation where two free groups of rank 2 have a non-finitely generated (of course, free) intersection. This does not contradict the Howson property for free groups, but rather indicates that one cannot embed H and H' simultaneously into a free subgroup of $\mathbb{Z} \times F_2$.

In the present section, we shall solve $\mathrm{SIP}(\mathbb{Z}^m \times F_n)$ and $\mathrm{CIP}(\mathbb{Z}^m \times F_n)$. The key point is Corollary 1.8: $H \cap H'$ is finitely generated if and only if $(H \cap H')\pi \leqslant F_n$ is finitely generated. Note that the group $H\pi \cap H'\pi$ is always finitely generated (by Howson property of F_n), but the inclusion $(H \cap H')\pi \leqslant H\pi \cap H'\pi$ is not (in general) an equality (for example, in $\mathbb{Z} \times F_2 = \langle t \mid \rangle \times \langle a, b \mid \rangle$, the subgroups $H = \langle t^2, ta^2 \rangle$ and $H' = \langle t^2, t^2a^3 \rangle$ satisfy $a^6 \in H\pi \cap H'\pi$ but $a^6 \notin (H \cap H')\pi$). This opens the possibility for $(H \cap H')\pi$, and so $H \cap H'$, to be non finitely generated, as is the case in the example from Observation 4.3.

Let us describe in detail the data involved in $\mathrm{CIP}(G)$ for $G = \mathbb{Z}^m \times F_n$. By Proposition 1.9, we can assume that the initial finitely generated subgroups $H, H' \leqslant G$ are given by respective bases i.e. by two sets of elements

$$\begin{aligned} E &= \{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m_1}}}, \mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n_1}}} u_{n_1}\}, \\ E' &= \{\mathbf{t}^{\mathbf{b'_1}}, \ldots, \mathbf{t}^{\mathbf{b'_{m_2}}}, \mathbf{t}^{\mathbf{a'_1}} u'_1, \ldots, \mathbf{t}^{\mathbf{a'_{n_2}}} u'_{n_2}\}, \end{aligned} \tag{4.1}$$

where $\{u_1, \ldots, u_{n_1}\}$ is a free basis of $H\pi \leqslant F_n$, $\{u'_1, \ldots, u'_{n_2}\}$ is a free basis of $H'\pi \leqslant F_n$, $\{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m_1}}}\}$ is an abelian basis of $H \cap \mathbb{Z}^m$, and $\{\mathbf{t}^{\mathbf{b'_1}}, \ldots, \mathbf{t}^{\mathbf{b'_{m_2}}}\}$ is an abelian basis of $H' \cap \mathbb{Z}^m$. Consider the subgroups $L = \langle \mathbf{b_1}, \ldots, \mathbf{b_{m_1}} \rangle \leqslant \mathbb{Z}^m$ and $L' = \langle \mathbf{b'_1}, \ldots, \mathbf{b'_{m_2}} \rangle \leqslant \mathbb{Z}^m$, and the matrices

$$\mathbf{A} = \begin{pmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{n_1}} \end{pmatrix} \in \mathcal{M}_{n_1 \times m}(\mathbb{Z}) \quad \text{and} \quad \mathbf{A}' = \begin{pmatrix} \mathbf{a'_1} \\ \vdots \\ \mathbf{a'_{n_2}} \end{pmatrix} \in \mathcal{M}_{n_2 \times m}(\mathbb{Z}).$$

We are also given two elements $g = \mathbf{t}^{\mathbf{a}} u$ and $g' = \mathbf{t}^{\mathbf{a}'} u'$ from G, and have to algorithmically decide whether the intersection $gH \cap g'H'$ is empty or not.

Before start describing the algorithm, note that $H\pi$ is a free group of rank n_1. Since $\{u_1, \ldots, u_{n_1}\}$ is a free basis of $H\pi$, every element $w \in H\pi$ can be written in a unique way as a word on the u_i's, say $w = \omega(u_1, \ldots, u_{n_1})$. Abelianizing this word, we get the abelianization map $\rho_1 \colon H\pi \twoheadrightarrow \mathbb{Z}^{n_1}$, $w \mapsto \boldsymbol{\omega}$ (not to be confused with the restriction to $H\pi$ of the ambient abelianization $F_n \twoheadrightarrow \mathbb{Z}^n$, which will have no role in this proof). Similarly, we define the morphism $\rho_2 \colon H'\pi \twoheadrightarrow \mathbb{Z}^{n_2}$.

With all this data given, note that $gH \cap g'H'$ is empty if and only if its projection to the free component is empty,

$$gH \cap g'H' = \varnothing \;\Leftrightarrow\; (gH \cap g'H')\pi = \varnothing;$$

so, it will be enough to study this last projection. And, since this projection contains precisely those elements from $(gH)\pi \cap (g'H')\pi = (u \cdot H\pi) \cap (u' \cdot H'\pi)$ having compatible abelian completions in $gH \cap g'H'$, a direct application of Lemma 1.13 gives the following result.

Lemma 4.4. *With the above notation, the projection $(gH \cap g'H')\pi$ consists precisely on those elements $v \in (u \cdot H\pi) \cap (u' \cdot H'\pi)$ such that*

$$N_v = (\mathbf{a} + \boldsymbol{\omega}\mathbf{A} + L) \cap (\mathbf{a}' + \boldsymbol{\omega}'\mathbf{A}' + L') \neq \varnothing, \tag{4.2}$$

where $\boldsymbol{\omega} = w\rho_1$ and $\boldsymbol{\omega}' = w'\rho_2$ are, respectively, the abelianizations of the abstract words $\omega \in F_{n_1}$ and $\omega' \in F_{n_2}$ expressing $w = u^{-1}v \in H\pi \leqslant F_n$ and $w' = u'^{-1}v \in H'\pi \leqslant F_n$ in terms of the free bases $\{u_1, \ldots, u_{n_1}\}$ and $\{u'_1, \ldots, u'_{n_2}\}$ (i.e. $u \cdot \omega(u_1, \ldots, u_{n_1}) = v = u' \cdot \omega'(u'_1, \ldots, u'_{n_2})$). That is,

$$(gH \cap g'H')\pi = \{v \in (u \cdot H\pi) \cap (u' \cdot H'\pi) \mid N_v \neq \varnothing\} \subseteq (u \cdot H\pi) \cap (u' \cdot H'\pi) \qquad \square$$

Theorem 4.5. *The Coset Intersection Problem for $\mathbb{Z}^m \times F_n$ is solvable.*

Proof. Let $G = \mathbb{Z}^m \times F_n$ be a finitely generated free-abelian times free group. Using the solution to $\mathrm{CIP}(F_n)$, we start by checking whether $(u \cdot H\pi) \cap (u' \cdot H'\pi)$ is empty or not. In the first case $(gH \cap g'H')\pi$, and so $gH \cap g'H'$, will also be empty and we are done. Otherwise, we can compute $v_0 \in F_n$ such that

$$(u \cdot H\pi) \cap (u' \cdot H'\pi) = v_0 \cdot (H\pi \cap H'\pi), \tag{4.3}$$

compute words $\omega_0 \in F_{n_1}$ and $\omega'_0 \in F_{n_2}$ such that $u \cdot \omega_0(u_1, \ldots, u_{n_1}) = v_0 = u' \cdot \omega'_0(u'_1, \ldots, u'_{n_2})$, and compute a free basis, $\{v_1, \ldots, v_{n_3}\}$, for $H\pi \cap H'\pi$ together with expressions of the v_i's in terms of the free bases for $H\pi$ and $H'\pi$, $v_i = \nu_i(u_1, \ldots, u_{n_1}) = \nu'_i(u'_1, \ldots, u'_{n_2})$, $i \in [n_3]$.

Let $\rho_3 \colon H\pi \cap H'\pi \twoheadrightarrow \mathbb{Z}^{n_3}$ be the corresponding abelianization map. Abelianizing the words ν_i and ν'_i, we can compute the rows of the matrices $\mathbf{P}$ and $\mathbf{P}'$ (of sizes $n_3 \times n_1$ and $n_3 \times n_2$, respectively) describing the abelianizations of the inclusion maps $H\pi \overset{\iota}{\hookleftarrow} H\pi \cap H'\pi \overset{\iota'}{\hookrightarrow} H'\pi$, see the central part of the diagram (4.4) below.

By (4.3), $u^{-1}v_0 \in H\pi$ and $u'^{-1}v_0 \in H'\pi$. So, left translation by $w_0 = u^{-1}v_0$ is a permutation of $H\pi$ (not a homomorphism, unless $w_0 = 1$), say $\lambda_{w_0} \colon H\pi \to H\pi$, $x \mapsto w_0 x = u^{-1}v_0 x$. Analogously, we have the left translation by $w'_0 = u'^{-1}v_0$, say $\lambda_{w'_0} \colon H'\pi \to H'\pi$, $x \mapsto w'_0 x = u'^{-1}v_0 x$. We include these translations in our

diagram:

$$
\begin{tikzcd}
 & & (H\cap H')\pi \arrow[d, equal] & & \\
H\pi \arrow[d, two heads, "\rho_1"'] \arrow[dr, phantom, "///"] & H\pi \arrow[l, "\lambda_{w_0}"'] \arrow[d, two heads, "\rho_1"'] \arrow[dr, phantom, "///"] & H\pi\cap H'\pi \arrow[l, hook', "\iota"'] \arrow[r, hook, "\iota'"] \arrow[d, two heads, "\rho_3"] \arrow[dr, phantom, "///"] & H'\pi \arrow[r, "\lambda_{w'_0}"] \arrow[d, two heads, "\rho_2"] \arrow[dr, phantom, "///"] & H'\pi \arrow[d, two heads, "\rho_2"] \\
\mathbb{Z}^{n_1} \arrow[drr, "\mathbf{A}"'] & \mathbb{Z}^{n_1} \arrow[l, "+\boldsymbol{\omega_0}"'] \arrow[dr, "\mathbf{A}"] & \mathbb{Z}^{n_3} \arrow[l, "\mathbf{P}"'] \arrow[r, "\mathbf{P'}"] & \mathbb{Z}^{n_2} \arrow[r, "+\boldsymbol{\omega'_0}"] \arrow[dl, "\mathbf{A'}"'] & \mathbb{Z}^{n_2} \arrow[dll, "\mathbf{A'}"] \\
 & & \mathbb{Z}^m & &
\end{tikzcd}
\tag{4.4}
$$

where $\boldsymbol{\omega_0} = w_0\rho_1 \in \mathbb{Z}^{n_1}$ and $\boldsymbol{\omega'_0} = w'_0\rho_2 \in \mathbb{Z}^{n_2}$ are the abelianizations of w_0 and w'_0 with respect to the free bases $\{u_1,\ldots,u_{n_1}\}$ and $\{u'_1,\ldots,u'_{n_2}\}$, respectively.

Now, for every $v \in (u\cdot H\pi)\cap(u'\cdot H'\pi)$, using Lemma 4.4 and the commutativity of the upper part of the above diagram, we have

$$
\begin{aligned}
N_v &= \big(\mathbf{a} + (u^{-1}v)\rho_1\mathbf{A} + L\big) \cap \big(\mathbf{a'} + (u'^{-1}v)\rho_2\mathbf{A'} + L'\big)\\
&= \big(\mathbf{a} + (v_0^{-1}v)\iota\lambda_{w_0}\rho_1\mathbf{A} + L\big) \cap \big(\mathbf{a'} + (v_0^{-1}v)\iota'\lambda_{w'_0}\rho_2\mathbf{A'} + L'\big)\\
&= \big(\mathbf{a} + (\boldsymbol{\omega_0} + (v_0^{-1}v)\rho_3\mathbf{P})\mathbf{A} + L\big) \cap \big(\mathbf{a'} + (\boldsymbol{\omega'_0} + (v_0^{-1}v)\rho_3\mathbf{P'})\mathbf{A'} + L'\big)\\
&= \big(\mathbf{a} + \boldsymbol{\omega_0}\mathbf{A} + (v_0^{-1}v)\rho_3\mathbf{PA} + L\big) \cap \big(\mathbf{a'} + \boldsymbol{\omega'_0}\mathbf{A'} + (v_0^{-1}v)\rho_3\mathbf{P'A'} + L'\big)\,.
\end{aligned}
$$

With this expression, we can characterize, in a computable way, which elements from $(u\cdot H\pi)\cap(u'\cdot H'\pi)$ do belong to $(gH\cap g'H')\pi$:

Lemma 4.6. *With the current notation we have*

$$(gH\cap g'H')\pi = M\rho_3^{-1}\lambda_{v_0} \subseteq (u\cdot H\pi)\cap(u'\cdot H'\pi), \tag{4.5}$$

where $M \subseteq \mathbb{Z}^{n_3}$ is the preimage by the linear mapping $\mathbf{PA}-\mathbf{P'A'}\colon \mathbb{Z}^{n_3}\to\mathbb{Z}^m$ of the linear variety

$$N = \mathbf{a'} - \mathbf{a} + \boldsymbol{\omega'_0}\mathbf{A'} - \boldsymbol{\omega_0}\mathbf{A} + (L+L') \subseteq \mathbb{Z}^m. \tag{4.6}$$

Proof. By Lemma 4.4, an element $v \in (u\cdot H\pi)\cap(u'\cdot H'\pi)$ belongs to $(gH\cap g'H')\pi$ if and only if $N_v \neq \varnothing$. That is, if and only if the vector $\mathbf{x} = (v_0^{-1}v)\rho_3 \in \mathbb{Z}^{n_3}$ satisfies that the two varieties $\mathbf{a} + \boldsymbol{\omega_0}\mathbf{A} + \mathbf{xPA} + L$ and $\mathbf{a'} + \boldsymbol{\omega'_0}\mathbf{A'} + \mathbf{xP'A'} + L'$ do intersect. But this happens if and only if the vector

$$\big(\mathbf{a} + \boldsymbol{\omega_0}\mathbf{A} + \mathbf{xPA}\big) - \big(\mathbf{a'} + \boldsymbol{\omega'_0}\mathbf{A'} + \mathbf{xP'A'}\big) = \mathbf{a} - \mathbf{a'} + \boldsymbol{\omega_0}\mathbf{A} - \boldsymbol{\omega'_0}\mathbf{A'} + \mathbf{x}(\mathbf{PA} - \mathbf{P'A'})$$

belongs to $L+L'$. That is, if and only if $\mathbf{x}(\mathbf{PA}-\mathbf{P'A'})$ belongs to N. Hence, v belongs to $(gH\cap g'H')\pi$ if and only if $\mathbf{x} = (v_0^{-1}v)\rho_3 \in M$, i.e. if and only if $v \in M\rho_3^{-1}\lambda_{v_0}$. $\square$

With all the data already computed, we explicitly have the variety N and, using standard linear algebra, we can compute M (which could be empty, because N may possibly be disjoint with the image of $\mathbf{PA}-\mathbf{P'A'}$). In this situation, the algorithmic decision on whether $gH\cap g'H'$ is empty or not is straightforward.

Lemma 4.7. *With the current notation, and assuming that $(u\cdot H\pi)\cap(u'\cdot H'\pi)\neq\varnothing$, the following are equivalent:*

(a) $gH\cap g'H'=\varnothing$,

(b) $(gH\cap g'H')\pi=\varnothing$,

(c) $M\rho_3^{-1}=\varnothing$,

(d) $M=\varnothing$,

(e) $N\cap\operatorname{Im}(\mathbf{PA}-\mathbf{P'A'})=\varnothing$. □

If $gH\cap g'H'=\varnothing$, we are done. Otherwise, $N\cap\operatorname{Im}(\mathbf{PA}-\mathbf{P'A'})\neq\varnothing$ and we can compute a vector $\mathbf{x}\in\mathbb{Z}^{n_3}$ such that $\mathbf{x}(\mathbf{PA}-\mathbf{P'A'})\in N$. Take now any preimage of $\mathbf{x}$ by ρ_3, for example $v_1^{x_1}\cdots v_{n_3}^{x_{n_3}}$ if $\mathbf{x}=(x_1,\ldots,x_{n_3})$, and by (4.5), $u''=v_0v_1^{x_1}\cdots v_{n_3}^{x_{n_3}}\in(gH\cap g'H')\pi$.

It only remains to find $\mathbf{a}''\in\mathbb{Z}^m$ such that $g''=\mathbf{t}^{\mathbf{a}''}u''\in gH\cap g'H'$. To do this, observe that $u''\in(gH\cap g'H')\pi$ implies the existence of a vector $\mathbf{a}''$ such that $\mathbf{t}^{\mathbf{a}''}u''\in\mathbf{t}^{\mathbf{a}}uH\cap\mathbf{t}^{\mathbf{a}'}u'H'$, i.e. such that $\mathbf{t}^{\mathbf{a}''-\mathbf{a}}u^{-1}u''\in H$ and $\mathbf{t}^{\mathbf{a}''-\mathbf{a}'}u'^{-1}u''\in H'$. In other words, there exists a vector $\mathbf{a}''\in\mathbb{Z}^m$ such that $\mathbf{a}''-\mathbf{a}\in\mathcal{C}_{u^{-1}u'',H}$ and $\mathbf{a}''-\mathbf{a}'\in\mathcal{C}_{u'^{-1}u'',H'}$. That is, the affine varieties $\mathbf{a}+\mathcal{C}_{u^{-1}u'',H}$ and $\mathbf{a}'+\mathcal{C}_{u'^{-1}u'',H'}$ do intersect. By Corollary 1.15, we can compute equations for these two varieties, and compute a vector in its intersection. This is the $\mathbf{a}''\in\mathbb{Z}^m$ we are looking for. □

The above argument applied to the case where $g=g'=1$ is giving us valuable information about the subgroup intersection $H\cap H'$; this will allow us to solve $\mathrm{SIP}(\mathbb{Z}^m\times F_n)$ as well. Note that, in this case, $\mathbf{a}=\mathbf{a}'=\mathbf{0}$, $u=u'=1$ and so, $v_0=1$, $w_0=w_0'=1$, and $\boldsymbol{\omega}_{\mathbf{0}}=\boldsymbol{\omega}'_{\mathbf{0}}=\mathbf{0}$.

Theorem 4.8. *The Subgroup Intersection Problem for $\mathbb{Z}^m\times F_n$ is solvable.*

Proof. Let $G=\mathbb{Z}^m\times F_n$ be a finitely generated free-abelian times free group. As in the proof of Theorem 4.5, we can assume that the initial finitely generated subgroups $H,H'\leqslant G$ are given by respective bases, i.e. by two sets of elements like in (4.1), $E=\{\mathbf{t}^{\mathbf{b_1}},\ldots,\mathbf{t}^{\mathbf{b_{m_1}}},\mathbf{t}^{\mathbf{a_1}}u_1,\ldots,\mathbf{t}^{\mathbf{a_{n_1}}}u_{n_1}\}$ and $E'=\{\mathbf{t}^{\mathbf{b}'_\mathbf{1}},\ldots,\mathbf{t}^{\mathbf{b}'_{\mathbf{m_2}}},\mathbf{t}^{\mathbf{a}'_\mathbf{1}}u'_1,\ldots,\mathbf{t}^{\mathbf{a}'_{\mathbf{n_2}}}u'_{n_2}\}$. Consider the subgroups $L,L'\leqslant\mathbb{Z}^m$ and the matrices $\mathbf{A}\in\mathcal{M}_{n_1\times m}(\mathbb{Z})$ and $\mathbf{A}'\in\mathcal{M}_{n_2\times m}(\mathbb{Z})$ as above. We shall algorithmically decide whether the intersection $H\cap H'$ is finitely generated or not and, in the affirmative case, shall compute a basis for $H\cap H'$.

Let us apply the algorithm from the proof of Theorem 4.5 to the cosets $1\cdot H$ and $1\cdot H'$; that is, take $g=g'=1$, i.e. $u=u'=1$ and $\mathbf{a}=\mathbf{a}'=\mathbf{0}$. Of course, $H\cap H'$

is not empty, and $v_0 = 1$ serves as an element in the intersection, $v_0 \in H \cap H'$. With this choice, the algorithm works with $w_0 = w'_0 = 1$ and $\boldsymbol{\omega_0} = \boldsymbol{\omega'_0} = \mathbf{0}$ (so, we can forget the two translation parts in diagram (4.4)). Lemma 4.6 tells us that $(H \cap H')\pi = M\rho_3^{-1} \leqslant H\pi \cap H'\pi$, where M is the preimage by the linear mapping $\mathbf{PA} - \mathbf{P'A'}: \mathbb{Z}^{n_3} \to \mathbb{Z}^{n_1}$ of the subspace $N = L + L' \leqslant \mathbb{Z}^m$. In this situation, the following lemma decides when is $H \cap H'$ finitely generated and when is not:

Lemma 4.9. *With the current notation, the following are equivalent:*

(a) *$H \cap H'$ is finitely generated,*

(b) *$(H \cap H')\pi$ is finitely generated,*

(c) *$M\rho_3^{-1}$ is either trivial or of finite index in $H\pi \cap H'\pi$,*

(d) *either $n_3 = 1$ and $M = \{\mathbf{0}\}$, or M is of finite index in $\mathbb{Z}^{n_3}$,*

(e) *either $n_3 = 1$ and $M = \{\mathbf{0}\}$, or $\operatorname{rk}(M) = n_3$.*

Proof. (a) $\Leftrightarrow$ (b) is in Corollary 1.8. (b) $\Leftrightarrow$ (c) comes from the well known fact (see, for example, [18] pags. 16-18) that, in the finitely generated free group $H\pi \cap H'\pi$, the subgroup $(H \cap H')\pi = M\rho_3^{-1}$ is normal and so, finitely generated if and only if it is either trivial or of finite index. But, by lemma 3.2 (ii), the index $[H\pi \cap H'\pi : M\rho_3^{-1}]$ is finite if and only if $[\mathbb{Z}^{n_3} : M]$ is finite; this gives (c) $\Leftrightarrow$ (d). The last equivalence is a basic fact in linear algebra. □

We have computed n_3 and an abelian basis for M. If $n_3 = 0$ we immediately deduce that $H \cap H'$ is finitely generated. If $n_3 = 1$ and $M = \{\mathbf{0}\}$ we also deduce that $H \cap H'$ is finitely generated. Otherwise, we check whether $\operatorname{rk}(M)$ equals n_3; if this is the case then again $H \cap H'$ is finitely generated; if not, $H \cap H'$ is infinitely generated.

It only remains to algorithmically compute a basis for $H \cap H'$, in case it is finitely generated. We know from (1.5) that

$$H \cap H' = \big((H \cap H') \cap \mathbb{Z}^m\big) \times (H \cap H')\pi\alpha,$$

where α is any splitting for $\pi_{|H \cap H'}: H \cap H' \twoheadrightarrow (H \cap H')\pi$; then we can easily get a basis of $H \cap H'$ by putting together a basis of each part. The strategy will be the following: first, we compute an abelian basis for

$$(H \cap H') \cap \mathbb{Z}^m = (H \cap \mathbb{Z}^m) \cap (H' \cap \mathbb{Z}^m) = L \cap L'$$

by just solving a system of linear equations. Second, we shall compute a free basis for $(H \cap H')\pi$. And finally, we will construct an explicit splitting α and will use it to get a free basis for $(H \cap H')\pi\alpha$. Putting together these two parts, we shall be done.

To compute a free basis for $(H \cap H')\pi$ note that, if $n_3 = 0$, or $n_3 = 1$ and $M = \{\mathbf{0}\}$, then $(H \cap H')\pi = 1$ and there is nothing to do. In the remaining case,

$\mathrm{rk}(M) = n_3 \geqslant 1$, $M\rho_3^{-1} = (H \cap H')\pi$ has finite index in $H\pi \cap H'\pi$, and so it is finitely generated. We give two alternative options to compute a free basis for it.

The subgroup M has finite index in $\mathbb{Z}^{n_3}$, and we can compute a system of coset representatives of $\mathbb{Z}^{n_3}$ modulo M,

$$\mathbb{Z}^{n_3} = M\mathbf{c_1} \sqcup \cdots \sqcup M\mathbf{c_d}$$

(see the beginning of Section 3). Now, being ρ_3 onto, and according to Lemma 3.2 (b), we can transfer the previous partition via ρ_3 to obtain a system of right coset representatives of $H\pi \cap H'\pi$ modulo $M\rho_3^{-1}$:

$$H\pi \cap H'\pi = (M\rho_3^{-1})z_1 \sqcup \cdots \sqcup (M\rho_3^{-1})z_d, \tag{4.7}$$

where we can take, for example, $z_i = v_1^{c_{i,1}} v_2^{c_{i,2}} \cdots v_{n_3}^{c_{i,n_3}} \in H\pi \cap H'\pi$, for each vector $\mathbf{c_i} = (c_{i,1}, c_{i,2}, \ldots, c_{i,n_3}) \in \mathbb{Z}^{n_3}$, $i \in [d]$. Now let us construct the core of the Schreier graph for $M\rho_3^{-1} = (H \cap H')\pi$ (with respect to $\{v_1, \ldots, v_{n_3}\}$, a free basis for $H\pi \cap H'\pi$), $\mathcal{S}(M\rho_3^{-1})$, in the following way: consider the graph with the cosets of (4.7) as vertices, and with no edge. Then, for every vertex $(M\rho_3^{-1})z_i$ and every letter v_j, add an edge labeled v_j from $(M\rho_3^{-1})z_i$ to $(M\rho_3^{-1})z_i v_j$, algorithmically identified among the available vertices by repeatedly using the *membership problem* for $M\rho_3^{-1}$ (note that we can do this by abelianizing the candidate and checking the defining equations for M). Once we have run over all i, j, we shall get the full graph $\mathcal{S}(M\rho_3^{-1})$, from which we can easily obtain a free basis for $(H \cap H')\pi$ in terms of $\{v_1, \ldots, v_{n_3}\}$.

Alternatively, let $\{\mathbf{m_1}, \ldots, \mathbf{m_{n_3}}\}$ be an abelian basis for M (which we already have from the previous construction), say $\mathbf{m_i} = (m_{i,1}, m_{i,2}, \ldots, m_{i,n_3}) \in \mathbb{Z}^{n_3}$, $i = 1, \ldots, n_3$, and consider the elements $x_i = v_1^{m_{i,1}} v_2^{m_{i,2}} \cdots v_{n_3}^{m_{i,n_3}} \in H\pi \cap H'\pi$. It is clear that $M\rho_3^{-1}$ is the subgroup of $H\pi \cap H'\pi$ generated by $x_1, \ldots, x_{n_3}$ and all the infinitely many commutators from elements in $H\pi \cap H'\pi$. But $M\rho_3^{-1}$ is finitely generated so, finitely many of those commutators will be enough. Enumerate all of them, $y_1, y_2, \ldots$ and keep computing the core $\mathcal{S}_j$ of the Schreier graph for the subgroup $\langle x_1, \ldots, x_{n_3}, y_1, \ldots, y_j \rangle$ for increasing j's until obtaining a complete graph with d vertices (i.e. until reaching a subgroup of index d). When this happens, we shall have computed the core of the Schreier graph for $M\rho_3^{-1} = (H \cap H')\pi$ (with respect to $\{v_1, \ldots, v_{n_3}\}$, a free basis of $H\pi \cap H'\pi$), from which we can easily find a free basis for $(H \cap H')\pi$, in terms of $\{v_1, \ldots, v_{n_3}\}$.

Finally, it remains to compute an explicit splitting α for $\pi_{|H\cap H'} \colon H \cap H' \twoheadrightarrow (H \cap H')\pi$. We have a free basis $\{z_1, \ldots, z_d\}$ for $(H \cap H')\pi$, in terms of $\{v_1, \ldots, v_{n_3}\}$; so, using the expressions $v_i = \nu_i(u_1, \ldots, u_{n_1})$ that we have from the beginning of the proof, we can get expressions $z_i = \eta_i(u_1, \ldots, u_{n_1})$. From here, $\eta_i(\mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n_1}}} u_{n_1}) = \mathbf{t}^{\mathbf{e_i}} z_i \in H$ and projects to z_i, so $\mathcal{C}_{z_i, H} = \mathbf{e_i} + L$ (see Corollary 1.15), $i \in [d]$. Similarly, we can get vectors $\mathbf{e_i'} \in \mathbb{Z}^m$ such that $\mathcal{C}_{z_i, H'} = \mathbf{e_i'} + L'$. Since, by construction, $\mathcal{C}_{z_i, H\cap H'} = \mathcal{C}_{z_i, H} \cap \mathcal{C}_{z_i, H'}$ is a non-empty affine variety in $\mathbb{Z}^m$ with direction $L \cap L'$, we can compute vectors $\mathbf{e_i''} \in \mathbb{Z}^m$ on it by just solving the corresponding systems of linear equations, $i \in [d]$. Now, $z_i \mapsto \mathbf{t}^{\mathbf{e_i''}} z_i$ is the desired splitting $H \cap H' \xleftarrow{\alpha} (H \cap H')\pi$, and $\{\mathbf{t}^{\mathbf{e_1''}} z_1, \ldots, \mathbf{t}^{\mathbf{e_d''}} z_d\}$ is the free basis for $(H \cap H')\pi\alpha$ we were looking for.

As mentioned above, putting together this free basis with the abelian basis we already have for $L \cap L'$, we get a basis for $H \cap H'$, concluding the proof. $\square$

Corollary 4.10. *Let H, H' be two free non-abelian subgroups of finite rank in $\mathbb{Z}^m \times F_n$. With the previous notation, the intersection $H \cap H'$ is finitely generated if and only if either $H \cap H' = 1$, or $\mathbf{PA} = \mathbf{P'A'}$.*

Proof. Under the conditions of the statement, we have $L = L' = \{\mathbf{0}\}$. Hence, $N = L + L' = \{\mathbf{0}\}$ and its preimage by $\mathbf{PA} - \mathbf{P'A'}$ is $M = \ker(\mathbf{PA} - \mathbf{P'A'}) \leqslant \mathbb{Z}^{n_3}$. Now, by Lemma 4.9, $H \cap H'$ is finitely generated if and only if either $(H \cap H')\pi = M\rho_3^{-1} = 1$, or $n_3 - \mathrm{rk}(Im(\mathbf{PA} - \mathbf{P'A'})) = \mathrm{rk}(M) = n_3$; that is, if and only if either $(H \cap H')\pi = 1$, or $\mathbf{PA} = \mathbf{P'A'}$. But, since $L = L' = \{\mathbf{0}\}$, $(H \cap H')\pi = 1$ if and only if $H \cap H' = 1$. $\square$

We consider the following two examples to illustrate the preceding algorithm.

Example 4.11*.* Let us analyze again the example given in the proof of Observation 4.3, under the light of the previous corollary. We considered in $\mathbb{Z} \times F_2 = \langle t \mid \rangle \times \langle a, b \mid \rangle$ the subgroups $H = \langle a, b \rangle$ and $H' = \langle ta, b \rangle$, both free non-abelian of rank 2. It is clear that $\mathbf{A} = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and $\mathbf{A'} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, while $H\pi = H'\pi = H\pi \cap H'\pi = F_2$; in particular, $n_3 = 2$ and $H \cap H' \neq 1$. In these circumstances, both inclusions $H\pi \hookleftarrow H\pi \cap H'\pi \hookrightarrow H'\pi$ are the identity maps, so $\mathbf{P} = \mathbf{P'} = \mathbf{1}$ is the 2×2 identity matrix and hence, $\mathbf{PA} = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \mathbf{P'A'}$. According to Corollary 4.10, this means that $H \cap H'$ is not finitely generated, as we had seen before.

Example 4.12*.* Consider two finitely generated subgroups $H, H' \leqslant F_n \leqslant \mathbb{Z}^m \times F_n$. In this case we have $\mathbf{A} = (\mathbf{0}) \in \mathcal{M}_{n_1,m}$ and $\mathbf{A'} = (\mathbf{0}) \in \mathcal{M}_{n_2,m}$ and so, $\mathbf{PA} = (\mathbf{0}) = \mathbf{P'A'}$. Thus, Corollary 4.10 just corroborates Howson's property for finitely generated free groups.

To finish this section, we present an application of Theorem 4.8 to a nice geometric problem. In the very recent paper [23], J. Sahattchieve studies quasi-convexity of subgroups of $\mathbb{Z}^m \times F_n$ with respect to the natural component-wise action of $\mathbb{Z}^m \times F_n$ on the product space, $\mathbb{R}^m \times T_n$, of the m-dimensional euclidean space and the regular $(2n)$-valent infinite tree T_n: a subgroup $H \leqslant \mathbb{Z}^m \times F_n$ is *quasi-convex* if the orbit Hp of some (and hence every) point $p \in \mathbb{R}^m \times T_n$ is a quasi-convex subset of $\mathbb{R}^m \times T_n$ (see [23] for more details). One of the results obtained is the following characterization:

Theorem 4.13 (Sahattchieve)**.** *Let H be a subgroup of $\mathbb{Z}^m \times F_n$. Then, H is quasi-convex if and only if H is either cyclic or virtually of the form $A \times B$, for some $A \leqslant \mathbb{Z}^m$ and $B \leqslant F_n$ being finitely generated. (In particular, quasi-convex subgroups are finitely generated.)*

Combining this with our Theorem 4.8, we can easily establish an algorithm to decide whether a given finitely generated subgroup of $\mathbb{Z}^m \times F_n$ is quasi-convex or not (with respect to the above mentioned action).

Corollary 4.14. *There is an algorithm which, given a finite list $w_1, \ldots, w_s$ of elements in $\mathbb{Z}^m \times F_n$, decides whether the subgroup $H = \langle w_1, \ldots, w_s \rangle$ is quasi-convex or not.*

Proof. First, apply Proposition 1.9 to compute a basis for H. If it contains only one element, then H is cyclic and we are done.

Otherwise (H is not cyclic) we can easily compute a free-abelian basis and a free basis for the respective projections $H\tau \leqslant \mathbb{Z}^m$ and $H\pi \leqslant F_n$. From the basis for H we can immediately extract a free-abelian basis for $\mathbb{Z}^m \cap H = H\tau \cap H$. And, using Theorem 4.8, we can decide whether $F_n \cap H = H\pi \cap H$ is finitely generated or not and, in the affirmative case, compute a free basis for it. Finally, we can decide whether $H\tau \cap H \leqslant_{\text{f.i.}} H\tau$ and $H\pi \cap H \leqslant_{\text{f.i.}} H\pi$ hold or not (applying the well known solutions to $\mathrm{FIP}(\mathbb{Z}^m)$ and $\mathrm{FIP}(F_{n'})$ or, alternatively, using the more general Theorem 3.4 above); note that if we detected that $H\pi \cap H$ is infinitely generated then it must automatically be of infinite index in $H\pi$ (which, of course, is finitely generated).

Now we claim that H is quasi-convex if and only if $H\tau \cap H \leqslant_{\text{f.i.}} H\tau$ and $H\pi \cap H \leqslant_{\text{f.i.}} H\pi$; this will conclude the proof.

For the implication to the right (and applying Theorem 4.13), assume that $A \times B \leqslant_{\text{f.i.}} H$ for some $A \leqslant \mathbb{Z}^m$ and $B \leqslant F_n$ being finitely generated. Applying τ and π we get $A \leqslant_{\text{f.i.}} H\tau$ and $B \leqslant_{\text{f.i.}} H\pi$, respectively (see Lemma 3.2 (i)). But $A \leqslant H\tau \cap H \leqslant H\tau$ and $B \leqslant H\pi \cap H \leqslant H\pi$ hence, $H\tau \cap H \leqslant_{\text{f.i.}} H\tau$ and $H\pi \cap H \leqslant_{\text{f.i.}} H\pi$.

For the implication to the left, assume $H\tau \cap H \leqslant_{\text{f.i.}} H\tau$ and $H\pi \cap H \leqslant_{\text{f.i.}} H\pi$ (and, in particular, $H\pi \cap H$ finitely generated). Take $A = H\tau \cap H \leqslant_{\text{f.i.}} H\tau \leqslant \mathbb{Z}^m$ and $B = H\pi \cap H \leqslant_{\text{f.i.}} H\pi \leqslant F_n$, and we get $A \times B \leqslant_{\text{f.i.}} H\tau \times H\pi$ (see Lemma 3.3). But H is in between, $A \times B \leqslant H \leqslant H\tau \times H\pi$, hence $A \times B \leqslant_{\text{f.i.}} H$ and, by Theorem 4.13, H is quasi-convex. □

5 Endomorphisms

In this section we will study the endomorphisms of a finitely generated free-abelian times free group $G = \mathbb{Z}^m \times F_n$ (with the notation from presentation (1.2)). Without loss of generality, we assume $n \neq 1$.

To clarify notation, we shall use lowercase Greek letters to denote endomorphisms of F_n, and uppercase Greek letters to denote endomorphisms of $G = \mathbb{Z}^m \times F_n$. The following proposition gives a description of how all endomorphisms of G look like.

Proposition 5.1. *Let $G = \mathbb{Z}^m \times F_n$ with $n \neq 1$. The following is a complete list of all endomorphisms of G:*

- **(I)** $\Psi_{\phi,\mathbf{Q},\mathbf{P}} = \mathbf{t}^{\mathbf{a}} u \mapsto \mathbf{t}^{\mathbf{aQ}+\mathbf{uP}} u\phi$, *where* $\phi \in \mathrm{End}(F_n)$, $\mathbf{Q} \in \mathcal{M}_m(\mathbb{Z})$, *and* $\mathbf{P} \in \mathcal{M}_{n\times m}(\mathbb{Z})$.

(II) $\Psi_{z,\mathbf{l},\mathbf{h},\mathbf{Q},\mathbf{P}} = \mathbf{t}^{\mathbf{a}}u \mapsto \mathbf{t}^{\mathbf{aQ}+\mathbf{uP}} z^{\mathbf{a}\mathbf{l}^{\intercal}+\mathbf{u}\mathbf{h}^{\intercal}}$, *where* $1 \neq z \in F_n$ *is not a proper power,* $\mathbf{Q} \in \mathcal{M}_m(\mathbb{Z})$, $\mathbf{P} \in \mathcal{M}_{n\times m}(\mathbb{Z})$, $\mathbf{0} \neq \mathbf{l} \in \mathbb{Z}^m$, *and* $\mathbf{h} \in \mathbb{Z}^n$.

(In both cases, $\mathbf{u} \in \mathbb{Z}^n$ *denotes the abelianization of the word* $u \in F_n$*.)*

Proof. It is straightforward to check that all maps of types (I) and (II) are, in fact, endomorphisms of G.

To see that this is the complete list of all of them, let $\Psi: G \to G$ be an arbitrary endomorphism of G. Looking at the normal form of the images of the x_i's and t_j's, we have

$$\Psi: \left\{ \begin{array}{rcl} x_i & \longmapsto & \mathbf{t}^{\mathbf{p_i}} w_i \\ t_j & \longmapsto & \mathbf{t}^{\mathbf{q_j}} z_j, \end{array} \right. \tag{5.1}$$

where $\mathbf{p_i}, \mathbf{q_j} \in \mathbb{Z}^m$ and w_i, $z_j \in F_n$, $i \in [n]$, $j \in [m]$. Let us distinguish two cases.

Case 1: $z_j = 1$ *for all* $j \in [m]$. Denoting ϕ the endomorphism of F_n given by $x_i \mapsto w_i$, and $\mathbf{P}$ and $\mathbf{Q}$ the following integral matrices (of sizes $n \times m$ and $m \times m$, respectively)

$$\mathbf{P} = \begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nm} \end{pmatrix} = \begin{pmatrix} \mathbf{p_1} \\ \vdots \\ \mathbf{p_n} \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} q_{11} & \cdots & q_{1m} \\ \vdots & \ddots & \vdots \\ q_{m1} & \cdots & q_{mm} \end{pmatrix} = \begin{pmatrix} \mathbf{q_1} \\ \vdots \\ \mathbf{q_m} \end{pmatrix},$$

we can write

$$\Psi: \left\{ \begin{array}{rcl} u & \longmapsto & \mathbf{t}^{\mathbf{uP}} u\phi \\ \mathbf{t}^{\mathbf{a}} & \longmapsto & \mathbf{t}^{\mathbf{aQ}}, \end{array} \right.$$

where $u \in F_n$ and $\mathbf{a} \in \mathbb{Z}^m$. So, $(\mathbf{t}^{\mathbf{a}}u)\Psi = \mathbf{t}^{\mathbf{aQ}+\mathbf{uP}} u\phi$ and Ψ equals $\Psi_{\phi,\mathbf{Q},\mathbf{P}}$ from type (I).

Case 2: $z_k \neq 1$ *for some* $k \in [m]$. For Ψ to be well defined, $\mathbf{t}^{\mathbf{p_i}} w_i$ and $\mathbf{t}^{\mathbf{q_j}} z_j$ must all commute with $\mathbf{t}^{\mathbf{q_k}} z_k$, and so w_i and z_j with $z_k \neq 1$, for all $i \in [n]$ and $j \in [m]$. This means that $w_i = z^{h_i}$, $z_j = z^{l_j}$ for some integers $h_i, l_j \in \mathbb{Z}$, $i \in [n]$, $j \in [m]$, with $l_k \neq 0$, and some $z \in F_n$ not being a proper power. Hence, $(\mathbf{t}^{\mathbf{a}}u)\Psi = (\mathbf{t}^{\mathbf{a}}\Psi)(u\Psi) = (\mathbf{t}^{\mathbf{aQ}} z^{\mathbf{a}\mathbf{l}^{\intercal}})(\mathbf{t}^{\mathbf{uP}} z^{\mathbf{u}\mathbf{h}^{\intercal}}) = \mathbf{t}^{\mathbf{aQ}+\mathbf{uP}} z^{\mathbf{a}\mathbf{l}^{\intercal}+\mathbf{u}\mathbf{h}^{\intercal}}$ and Ψ equals $\Psi_{z,\mathbf{l},\mathbf{h},\mathbf{Q},\mathbf{P}}$ from type (II).

This completes the proof. □

Note that if $n = 0$ then type (I) and type (II) endomorphisms do coincide. Otherwise, type (II) endomorphisms will be seen to be neither injective nor surjective. The following proposition gives a quite natural characterization of which endomorphisms of type (I) are injective, and which are surjective. It is important to note that the matrix $\mathbf{P}$ plays absolutely no role in this matter.

Proposition 5.2. *Let* Ψ *be an endomorphism of* $G = \mathbb{Z}^m \times F_n$, *with* $n \geqslant 2$. *Then,*

(i) Ψ *is a monomorphism if and only if it is of type (I),* $\Psi = \Psi_{\phi,\mathbf{Q},\mathbf{P}}$, *with* ϕ *a monomorphism of* F_n, *and* $\det(\mathbf{Q}) \neq 0$,

(ii) Ψ *is an epimorphism if and only if it is of type (I),* $\Psi = \Psi_{\phi,\mathbf{Q},\mathbf{P}}$*, with* ϕ *an epimorphism of* F_n*, and* $\det(\mathbf{Q}) = \pm 1$.

(iii) Ψ *is an automorphism if and only if it is of type (I),* $\Psi = \Psi_{\phi,\mathbf{Q},\mathbf{P}}$*, with* $\phi \in \mathrm{Aut}(F_n)$ *and* $\mathbf{Q} \in GL_m(\mathbb{Z})$*; in this case,* $(\Psi_{\phi,\mathbf{Q},\mathbf{P}})^{-1} = \Psi_{\phi^{-1},\mathbf{Q}^{-1},-\mathbf{M}^{-1}\mathbf{P}\mathbf{Q}^{-1}}$*, where* $\mathbf{M} \in \mathrm{GL}_n(\mathbb{Z})$ *is the abelianization of* ϕ.

Proof. (i). Suppose that Ψ is injective. Then Ψ can not be of type (II) since, if it were, the commutator of any two elements in F_n $(n \geqslant 2)$ would be in the kernel of Ψ. Hence, $\Psi = \Psi_{\phi,\mathbf{Q},\mathbf{P}}$ for some $\phi \in \mathrm{End}(F_n)$, $\mathbf{Q} \in \mathcal{M}_m(\mathbb{Z})$, and $\mathbf{P} \in \mathcal{M}_{n\times m}(\mathbb{Z})$. Since $\mathbf{t}^{\mathbf{a}}\Psi = \mathbf{t}^{\mathbf{aQ}}$, the injectivity of Ψ implies that of $\mathbf{a} \mapsto \mathbf{aQ}$; hence, $\det(\mathbf{Q}) \neq 0$. Finally, in order to prove the injectivity of ϕ, let $u \in F_n$ with $u\phi = 1$. Note that the endomorphism of $\mathbb{Q}^m$ given by $\mathbf{Q}$ is invertible so, in particular, there exist $\mathbf{v} \in \mathbb{Q}^m$ such that $\mathbf{vQ} = \mathbf{uP}$; write $\mathbf{v} = \frac{1}{b}\mathbf{a}$ for some $\mathbf{a} \in \mathbb{Z}^m$ and $b \in \mathbb{Z}$, $b \neq 0$, and we have $\mathbf{aQ} = b\,\mathbf{vQ} = b\,\mathbf{uP}$; thus, $(\mathbf{t}^{\mathbf{a}}u^{-b})\Psi = \mathbf{t}^{\mathbf{aQ}}(\mathbf{t}^{\mathbf{uP}}1)^{-b} = \mathbf{t}^{\mathbf{aQ}-b\mathbf{uP}} = \mathbf{t}^{\mathbf{0}} = 1$. Hence, $\mathbf{t}^{\mathbf{a}}u^{-b} = 1$ and so, $u = 1$.

Conversely, let $\Psi = \Psi_{\phi,\mathbf{Q},\mathbf{P}}$ be of type (I), with ϕ a monomorphism of F_n and $\det(\mathbf{Q}) \neq \mathbf{0}$, and let $\mathbf{t}^{\mathbf{a}}u \in G$ be such that $1 = (\mathbf{t}^{\mathbf{a}}u)\Psi = \mathbf{t}^{\mathbf{aQ}+\mathbf{uP}}\,u\phi$. Then, $u\phi = 1$ and so, $u = 1$; and $\mathbf{0} = \mathbf{aQ} + \mathbf{uP} = \mathbf{aQ}$ and so, $\mathbf{a} = \mathbf{0}$. Hence, Ψ is injective.

(ii). Suppose that Ψ is onto. Since the image of an endomorphism of type (II) followed by the projection π onto F_n, $n \geqslant 2$, is contained in $\langle z \rangle$ (and so is cyclic), Ψ cannot be of type (II). Hence, $\Psi = \Psi_{\phi,\mathbf{Q},\mathbf{P}}$ for some $\phi \in \mathrm{End}(F_n)$, $\mathbf{Q} \in \mathcal{M}_m(\mathbb{Z})$, and $\mathbf{P} \in \mathcal{M}_{n\times m}(\mathbb{Z})$. Given $v \in F_n \leqslant G$ there must be $\mathbf{t}^{\mathbf{a}}u \in G$ such that $(\mathbf{t}^{\mathbf{a}}u)\Psi = v$ and so $u\phi = v$. Thus $\phi\colon F_n \to F_n$ is onto. On the other hand, for every $j \in [m]$, let $\boldsymbol{\delta}_{\mathbf{j}}$ be the canonical vector of $\mathbb{Z}^m$ with 1 at coordinate j, and let $\mathbf{t}^{\mathbf{b_j}}u_j \in G$ be a pre-image by Ψ of $t_j = \mathbf{t}^{\boldsymbol{\delta}_{\mathbf{j}}}$. We have $(\mathbf{t}^{\mathbf{b_j}}u_j)\Psi = \mathbf{t}^{\boldsymbol{\delta}_{\mathbf{j}}}$, i.e. $u_j\phi = 1$, $\mathbf{u_j} = \mathbf{0}$ and $\mathbf{b_j Q} = \mathbf{b_j Q} + \mathbf{u_j P} = \boldsymbol{\delta}_{\mathbf{j}}$. This means that the matrix $\mathbf{B}$ with rows $\mathbf{b_j}$ satisfies $\mathbf{BQ} = \mathbf{I_m}$ and thus, $\det(\mathbf{Q}) = \pm 1$.

Conversely, let $\Psi = \Psi_{\phi,\mathbf{Q},\mathbf{P}}$ be of type (I), with ϕ being an epimorphism of F_n and $\det(\mathbf{Q}) = \pm 1$. By the hopfianity of F_n, $\phi \in \mathrm{Aut}(F_n)$ and we can consider $\Upsilon = \Psi_{\phi^{-1},\mathbf{Q}^{-1},-\mathbf{M}^{-1}\mathbf{P}\mathbf{Q}^{-1}}$, where $\mathbf{M} \in GL_n(\mathbb{Z})$ is the abelianization of ϕ. For every $\mathbf{t}^{\mathbf{a}}u \in G$, we have

$$(\mathbf{t}^{\mathbf{a}}u)\Upsilon\Psi = \big(\mathbf{t}^{\mathbf{a}\mathbf{Q}^{-1}-\mathbf{u}\mathbf{M}^{-1}\mathbf{P}\mathbf{Q}^{-1}}(u\phi^{-1})\big)\Psi = \mathbf{t}^{\mathbf{a}-\mathbf{u}\mathbf{M}^{-1}\mathbf{P}+\mathbf{u}\mathbf{M}^{-1}\mathbf{P}}u = \mathbf{t}^{\mathbf{a}}u.$$

Hence, Ψ is onto.

(iii). The equivalence is a direct consequence of (i) and (ii). To see the actual value of Ψ^{-1} it remains to compute the composition in the reverse order:

$$(\mathbf{t}^{\mathbf{a}}u)\Psi\Upsilon = \big(\mathbf{t}^{\mathbf{aQ}+\mathbf{uP}}(u\phi)\big)\Upsilon = \mathbf{t}^{\mathbf{a}+\mathbf{u}\mathbf{P}\mathbf{Q}^{-1}-\mathbf{u}\mathbf{M}\mathbf{M}^{-1}\mathbf{P}\mathbf{Q}^{-1}}u = \mathbf{t}^{\mathbf{a}}u. \qquad \square$$

Immediately from these characterizations for an endomorphism to be mono, epi or auto, we have the following corollary.

Corollary 5.3. $\mathbb{Z}^m \times F_n$ *is hopfian and not cohopfian.* $\square$

The hopfianity of free-abelian times free groups was already known as part of a bigger result: in [13] and [15] it was shown that finitely generated partially commutative groups (this includes groups of the form $G = \mathbb{Z}^m \times F_n$) are residually finite and so, hophian. However, our proof is more direct and explicit in the sense of giving complete characterizations of the injectivity and surjectivity of a given endomorphism of G. We remark that, despite it could seem reasonable, the hophianity of $\mathbb{Z}^m \times F_n$ does not follow directly from that of free-abelian and free groups (both very well known): in [26], the author constructs a direct product of two hophian groups which is *not* hophian.

For later use, next lemma summarizes how to operate type (I) endomorphisms (compose, invert and take a power); it can be easily proved by following routine computations. The reader can easily find similar equations for the composition of two type (II) endomorphisms, or one of each (we do not include them here because they will not be necessary for the rest of the paper).

Lemma 5.4. *Let $\Psi_{\phi,\mathbf{Q},\mathbf{P}}$ and $\Psi_{\phi',\mathbf{Q}',\mathbf{P}'}$ be two type (I) endomorphisms of $G = \mathbb{Z}^m \times F_n$, $n \neq 1$, and denote by $\mathbf{M} \in \mathcal{M}_n(\mathbb{Z})$ the (matrix of the) abelianization of $\phi \in \mathrm{End}(F_n)$. Then,*

(i) $\Psi_{\phi,\mathbf{Q},\mathbf{P}} \cdot \Psi_{\phi',\mathbf{Q}',\mathbf{P}'} = \Psi_{\phi\phi',\mathbf{QQ}',\mathbf{PQ}'+\mathbf{MP}'}$,

(ii) *for all $k \geqslant 1$, $(\Psi_{\phi,\mathbf{Q},\mathbf{P}})^k = \Psi_{\phi^k,\mathbf{Q}^k,\mathbf{P_k}}$, where $\mathbf{P_k} = \sum_{i=1}^{k} \mathbf{M}^{i-1}\mathbf{P}\mathbf{Q}^{k-i}$,*

(iii) *$\Psi_{\phi,\mathbf{Q},\mathbf{P}}$ is invertible if and only if $\phi \in \mathrm{Aut}(F_n)$ and $\mathbf{Q} \in \mathrm{GL}_m(\mathbb{Z})$; in this case, $(\Psi_{\phi,\mathbf{Q},\mathbf{P}})^{-1} = \Psi_{\phi^{-1},\mathbf{Q}^{-1},-\mathbf{M}^{-1}\mathbf{P}\mathbf{Q}^{-1}}$.*

(iv) *For every $\mathbf{a} \in \mathbb{Z}^m$ and $u \in F_n$, the right conjugation by $\mathbf{t^a}u$ is $\Gamma_{\mathbf{t^a}u} = \Psi_{\gamma_u,\mathbf{I_m},\mathbf{0}}$, where γ_u is the right conjugation by u in F_n, $v \mapsto u^{-1}vu$, $\mathbf{I_m}$ is the identity matrix of size m, and $\mathbf{0}$ is the zero matrix of size $n \times m$.* □

In the rest of the section, we shall use this information to derive the structure of $\mathrm{Aut}(G)$, where $G = \mathbb{Z}^m \times F_n$, $m \geqslant 1$, $n \geqslant 2$.

Theorem 5.5. *For $G = \mathbb{Z}^m \times F_n$, with $m \geqslant 1$ and $n \geqslant 2$, the group $\mathrm{Aut}(G)$ is isomorphic to the semidirect product $\mathcal{M}_{n\times m}(\mathbb{Z}) \rtimes (\mathrm{Aut}(F_n) \times \mathrm{GL}_m(\mathbb{Z}))$ with respect to the natural action. In particular, $\mathrm{Aut}(G)$ is finitely presented.*

Proof. First or all note that, for every ϕ, $\phi' \in \mathrm{Aut}(F_n)$, every $\mathbf{Q}$, $\mathbf{Q}' \in \mathrm{GL}_m(\mathbb{Z})$, and every $\mathbf{P}, \mathbf{P}' \in \mathcal{M}_{n\times m}(\mathbb{Z})$, we have

$$\Psi_{\phi,\mathbf{I_m},\mathbf{0}} \cdot \Psi_{\phi',\mathbf{I_m},\mathbf{0}} = \Psi_{\phi\phi',\mathbf{I_m},\mathbf{0}},$$

$$\Psi_{I_n,\mathbf{Q},\mathbf{0}} \cdot \Psi_{I_n,\mathbf{Q}',\mathbf{0}} = \Psi_{I_n,\mathbf{QQ}',\mathbf{0}}$$

$$\Psi_{I_n,\mathbf{I_m},\mathbf{P}} \cdot \Psi_{I_n,\mathbf{I_m},\mathbf{P}'} = \Psi_{I_n,\mathbf{I_m},\mathbf{P}+\mathbf{P}'}.$$

Hence, the three groups $\mathrm{Aut}(F_n)$, $\mathrm{GL}_m(\mathbb{Z})$, and $\mathcal{M}_{n\times m}(\mathbb{Z})$ (this last one with the addition of matrices), are all subgroups of $\mathrm{Aut}(G)$ via the three natural inclusions: $\phi \mapsto \Psi_{\phi,\mathbf{I_m},\mathbf{0}}$, $\mathbf{Q} \mapsto \Psi_{I_n,\mathbf{Q},\mathbf{0}}$, and $\mathbf{P} \mapsto \Psi_{I_n,\mathbf{I_m},\mathbf{P}}$, respectively. Furthermore, for

every $\phi \in \mathrm{Aut}(F_n)$ and every $\mathbf{Q} \in \mathrm{GL}_m(\mathbb{Z})$, it is clear that $\Psi_{\phi,\mathbf{I_m},\mathbf{0}} \cdot \Psi_{I_n,\mathbf{Q},\mathbf{0}} = \Psi_{I_n,\mathbf{Q},\mathbf{0}} \cdot \Psi_{\phi,\mathbf{I_m},\mathbf{0}}$; hence $\mathrm{Aut}(F_n) \times \mathrm{GL}_m(\mathbb{Z})$ is a subgroup of $\mathrm{Aut}(G)$ in the natural way.

On the other hand, for every $\phi \in \mathrm{Aut}(F_n)$, every $\mathbf{Q} \in \mathrm{GL}_m(\mathbb{Z})$, and every $\mathbf{P} \in \mathcal{M}_{n\times m}(\mathbb{Z})$, we have

$$(\Psi_{\phi,\mathbf{I_m},\mathbf{0}})^{-1} \cdot \Psi_{I_n,\mathbf{I_m},\mathbf{P}} \cdot \Psi_{\phi,\mathbf{I_m},\mathbf{0}} = \Psi_{\phi^{-1},\mathbf{I_m},\mathbf{0}} \cdot \Psi_{\phi,\mathbf{I_m},\mathbf{P}} = \Psi_{I_n,\mathbf{I_m},\mathbf{M^{-1}P}}, \tag{5.2}$$

where $\mathbf{M} \in \mathrm{GL}_n(\mathbb{Z})$ is the abelianization of ϕ, and

$$(\Psi_{I_n,\mathbf{Q},\mathbf{0}})^{-1} \cdot \Psi_{I_n,\mathbf{I_m},\mathbf{P}} \cdot \Psi_{I_n,\mathbf{Q},\mathbf{0}} = \Psi_{I_n,\mathbf{Q^{-1}},\mathbf{0}} \cdot \Psi_{I_n,\mathbf{Q},\mathbf{PQ}} = \Psi_{I_n,\mathbf{I_m},\mathbf{PQ}}. \tag{5.3}$$

In particular, $\mathcal{M}_{n\times m}(\mathbb{Z})$ is a normal subgroup of $\mathrm{Aut}(G)$. But $\mathrm{Aut}(F_n)$, $\mathrm{GL}_m(\mathbb{Z})$ and $\mathcal{M}_{n\times m}(\mathbb{Z})$ altogether generated the whole $\mathrm{Aut}(G)$, as can be seen with the equality

$$\Psi_{\phi,\mathbf{Q},\mathbf{P}} = \Psi_{I_n,\mathbf{I_m},\mathbf{PQ^{-1}}} \cdot \Psi_{I_n,\mathbf{Q},\mathbf{0}} \cdot \Psi_{\phi,\mathbf{I_m},\mathbf{0}}. \tag{5.4}$$

Thus, $\mathrm{Aut}(G)$ is isomorphic to the semidirect product $\mathcal{M}_{n\times m}(\mathbb{Z}) \rtimes (\mathrm{Aut}(F_n) \times \mathrm{GL}_m(\mathbb{Z}))$, with the action of $\mathrm{Aut}(F_n) \times \mathrm{GL}_m(\mathbb{Z})$ on $\mathcal{M}_{n\times m}(\mathbb{Z})$ given by equations (5.2) and (5.3).

But it is well known that these three groups are finitely presented: $\mathcal{M}_{n\times m}(\mathbb{Z}) \simeq \mathbb{Z}^{nm}$ is free-abelian generated by canonical matrices (with zeroes everywhere except for one position where there is a 1), $\mathrm{GL}_m(\mathbb{Z})$ is generated by elementary matrices, and $\mathrm{Aut}(F_n)$ is generated, for example, by the Nielsen automorphisms (see [18] for details and full finite presentations). Therefore, $\mathrm{Aut}(G)$ is also finitely presented (and one can easily obtain a presentation of $\mathrm{Aut}(G)$ by taking together the generators for $\mathcal{M}_{n\times m}(\mathbb{Z})$, $\mathrm{Aut}(F_n)$ and $\mathrm{GL}_m(\mathbb{Z})$, and putting as relations those of each of $\mathcal{M}_{n\times m}(\mathbb{Z})$, $\mathrm{Aut}(F_n)$ and $\mathrm{GL}_m(\mathbb{Z})$, together with the commutators of all generators from $\mathrm{Aut}(F_n)$ with all generators from $\mathrm{GL}_m(\mathbb{Z})$, and with the conjugacy relations describing the action of $\mathrm{Aut}(F_n) \times \mathrm{GL}_m(\mathbb{Z})$ on $\mathcal{M}_{n\times m}(\mathbb{Z})$ analyzed above). □

Finite presentability of $\mathrm{Aut}(G)$ was previously known as a particular case of a more general result: in [17], M. Laurence gave a finite family of generators for the group of automorphisms of any finitely generated partially commutative group, in terms of the underlying graph. It turns out that, when particularizing this to free-abelian times free groups, Laurence's generating set for $\mathrm{Aut}(G)$ is essentially the same as the one obtained here, after deleting some obvious redundancy. Later, in [9], M. Day builts a kind of peak reduction for such groups, from which he deduces finite presentation for its group of automorphisms. However, our Theorem 5.5 is better in the sense that it provides the explicit structure of the automorphism group of a free-abelian times free group.

6 The subgroup fixed by an endomorphism

In this section we shall study when the subgroup fixed by an endomorphism of $\mathbb{Z}^m \times F_n$ is finitely generated and, in this case, we shall consider the problem

of algorithmically computing a basis for it. We will consider the following two problems.

Problem 6.1 (**Fixed Point Problem,** $\mathrm{FPP}_{\mathbf{a}}(G)$)**.** *Given an automorphism* Ψ *of* G *(by the images of the generators), decide whether* $\operatorname{Fix}\Psi$ *is finitely generated and, if so, compute a set of generators for it.*

Problem 6.2 (**Fixed Point Problem,** $\mathrm{FPP}_{\mathbf{e}}(G)$)**.** *Given an endomorphism* Ψ *of* G *(by the images of the generators), decide whether* $\operatorname{Fix}\Psi$ *is finitely generated and, if so, compute a set of generators for it.*

Of course, the fixed point subgroup of an arbitrary endomorphism of $\mathbb{Z}^m$ is finitely generated, and the problems $\mathrm{FPP}_{\mathbf{e}}(\mathbb{Z}^m)$ and $\mathrm{FPP}_{\mathbf{a}}(\mathbb{Z}^m)$ are clearly solvable, just reducing to solve the corresponding systems of linear equations.

Again, the case of free groups is much more complicated. Gersten showed in [11] that $\operatorname{rk}(\operatorname{Fix}\phi) < \infty$ for every automorphism $\phi \in \operatorname{Aut}(F_n)$, and Goldstein and Turner [12] extended this result to arbitrary endomorphisms of F_n.

About computability, O. Maslakova published [21] in 2003, giving an algorithm to compute a free basis for $\operatorname{Fix}\phi$, where $\phi \in \operatorname{Aut}(F_n)$. After its publication, the arguments were found to be incorrect. An attempt to fix them and provide a correct solution to $\mathrm{FPP}_{\mathbf{a}}(F_n)$ has been recently made by O. Bogopolski and O. Maslakova in the preprint [6] not yet published (see the beginning of page 3); here, the arguments are quite involved and difficult, making strong and deep use of the theory of train tracks. It is worth mentioning at this point that, this problem was previously solved in some special cases with much simpler arguments and algorithms (see, for example Cohen and Lustig [20] for positive automorphisms, Turner [25] for special irreducible automorphisms, and Bogopolski [3] for the case $n = 2$). On the other hand, the problem $\mathrm{FPP}_{\mathbf{e}}(F_n)$ remains still open in general.

When one moves to free-abelian times free groups, the situation is even more involved. Similar to what happens with respect to the Howson property, $\operatorname{Fix}\Psi$ need not be finitely generated for $\Psi \in \operatorname{Aut}(\mathbb{Z} \times F_2)$, and essentially the same example from Observation 4.3 can be recycled here: consider the type (I) automorphism Ψ given by $a \mapsto ta$, $b \mapsto b$, $t \mapsto t$; clearly, $t^r w(a,b) \mapsto t^{r+|w|_a} w(a,b)$ and so,

$$\operatorname{Fix}\Psi = \{t^r w(a,b) \mid |w|_a = 0\} = \langle\!\langle t,\, b\rangle\!\rangle = \langle t, a^{-k} b a^k \ \ (k \in \mathbb{Z})\rangle$$

is not finitely generated.

In the present section we shall analyze how is the fixed point subgroup of an endomorphism of a free-abelian times free group, and we shall give an explicit characterization on when is it finitely generated. In the case it is, we shall also consider the computability of a finite basis for the fixed subgroup, and will solve the problems $\mathrm{FPP}_{\mathbf{a}}(\mathbb{Z}^m \times F_n)$ and $\mathrm{FPP}_{\mathbf{e}}(\mathbb{Z}^m \times F_n)$ modulo the corresponding problems for free groups, $\mathrm{FPP}_{\mathbf{a}}(F_n)$ and $\mathrm{FPP}_{\mathbf{e}}(F_n)$. (Our arguments descend directly from $\operatorname{End}(\mathbb{Z}^m \times F_n)$ to $\operatorname{End}(F_n)$, in such a way that any partial solution

to the free problems can be used to give the corresponding partial solution to the free-abelian times free problems, see Proposition 6.6 below.)

Let us distinguish the two types of endomorphisms according to Proposition 5.1 (and starting with the easier type (II) ones).

Proposition 6.3. *Let $G = \mathbb{Z}^m \times F_n$ with $n \neq 1$, and consider a type (II) endomorphism Ψ, namely*

$$\Psi = \Psi_{z,\mathbf{l},\mathbf{h},\mathbf{Q},\mathbf{P}}\colon \mathbf{t}^{\mathbf{a}} u \mapsto \mathbf{t}^{\mathbf{aQ}+\mathbf{uP}} z^{\mathbf{a}\mathbf{l}^{\top}+\mathbf{u}\mathbf{h}^{\top}},$$

where $1 \neq z \in F_n$ is not a proper power, $\mathbf{Q} \in \mathcal{M}_m(\mathbb{Z})$, $\mathbf{P} \in \mathcal{M}_{n\times m}(\mathbb{Z})$, $\mathbf{0} \neq \mathbf{l} \in \mathbb{Z}^m$, and $\mathbf{h} \in \mathbb{Z}^n$. Then, $\operatorname{Fix}\Psi$ is finitely generated, and a basis for $\operatorname{Fix}\Psi$ is algorithmically computable.

Proof. First note that $\operatorname{Im}\Psi$ is an abelian subgroup of $\mathbb{Z}^m \times F_n$. Then, by Corollary 1.7, it must be isomorphic to $\mathbb{Z}^{m'}$ for a certain $m' \leqslant m+1$. Therefore, $\operatorname{Fix}\Psi \leqslant \operatorname{Im}(\Psi)$ is isomorphic to a subgroup of $\mathbb{Z}^{m'}$, and thus finitely generated.

According to the definition, an element $\mathbf{t}^{\mathbf{a}} u$ is fixed by Ψ if and only if $\mathbf{t}^{\mathbf{aQ}+\mathbf{uP}} z^{\mathbf{a}\mathbf{l}^{\top}+\mathbf{u}\mathbf{h}^{\top}} = \mathbf{t}^{\mathbf{a}} u$. For this to be satisfied, u must be a power of z, say $u = z^r$ for certain $r \in \mathbb{Z}$, and abelianizing we get $\mathbf{u} = r\mathbf{z}$, and the system of equations

$$\left.\begin{aligned} \mathbf{a}\mathbf{l}^{\top} + r\mathbf{z}\mathbf{h}^{\top} &= r \\ \mathbf{a}(\mathbf{I_m} - \mathbf{Q}) &= r\mathbf{z}\mathbf{P} \end{aligned}\right\} \tag{6.1}$$

whose set $\mathcal{S}$ of integer solutions $(\mathbf{a}, r) \in \mathbb{Z}^{m+1}$ describe precisely the subgroup of fixed points by Ψ:

$$\operatorname{Fix}\Psi = \{\mathbf{t}^{\mathbf{a}} z^r \mid (\mathbf{a}, r) \in \mathcal{S}\}.$$

By solving (6.1), we get the desired basis for $\operatorname{Fix}\Psi$. The proof is complete. □

Theorem 6.4. *Let $G = \mathbb{Z}^m \times F_n$ with $n \neq 1$, and consider a type (I) endomorphism Ψ, namely*

$$\Psi = \Psi_{\phi,\mathbf{Q},\mathbf{P}}\colon \mathbf{t}^{\mathbf{a}} u \mapsto \mathbf{t}^{\mathbf{aQ}+\mathbf{uP}}\, u\phi,$$

where $\phi \in \operatorname{End}(F_n)$, $\mathbf{Q} \in \mathcal{M}_m(\mathbb{Z})$, and $\mathbf{P} \in \mathcal{M}_{n\times m}(\mathbb{Z})$. Let $N = \operatorname{Im}(\mathbf{I_m} - \mathbf{Q}) \cap \operatorname{Im}\mathbf{P}'$, where $\mathbf{P}'$ is the restriction of $\mathbf{P}\colon \mathbb{Z}^n \to \mathbb{Z}^m$ to $(\operatorname{Fix}\phi)\rho$, the image of $\operatorname{Fix}\phi \leqslant F_n$ under the global abelianization $\rho\colon F_n \twoheadrightarrow \mathbb{Z}^n$. Then, $\operatorname{Fix}\Psi$ is finitely generated if and only if one of the following happens: (i) *$\operatorname{Fix}\phi = 1$;* (ii) *$\operatorname{Fix}\phi$ is cyclic, $(\operatorname{Fix}\phi)\rho \neq \{\mathbf{0}\}$, and $N\mathbf{P}'^{-1} = \{\mathbf{0}\}$; or* (iii) *$\operatorname{rk}(N) = \operatorname{rk}(\operatorname{Im}\mathbf{P}')$.*

Proof. An element $\mathbf{t}^{\mathbf{a}} u$ is fixed by Ψ if and only if $\mathbf{t}^{\mathbf{aQ}+\mathbf{uP}} u\phi = \mathbf{t}^{\mathbf{a}} u$, i.e. if and only if

$$\left.\begin{aligned} u\phi &= u \\ \mathbf{a}(\mathbf{I_m} - \mathbf{Q}) &= \mathbf{uP} \end{aligned}\right\}$$

That is,

$$\operatorname{Fix}\Psi = \{\mathbf{t}^{\mathbf{a}} u \in G \mid u \in \operatorname{Fix}\phi \text{ and } \mathbf{a}(\mathbf{I_m} - \mathbf{Q}) = \mathbf{uP}\}, \tag{6.2}$$

where $\mathbf{u} = u\rho$, and $\rho\colon F_n \twoheadrightarrow \mathbb{Z}^n$ is the abelianization map. As we have seen in Corollary 1.8, $\mathrm{Fix}\,\Psi$ is finitely generated if and only if its projection to the free part

$$(\mathrm{Fix}\,\Psi)\pi = \mathrm{Fix}\,\phi \cap \{u \in F_n \mid \mathbf{uP} \in \mathrm{Im}(\mathbf{I_m} - \mathbf{Q})\} \tag{6.3}$$

is so. Now (identifying integral matrices $\mathbf{A}$ with the corresponding linear mapping $\mathbf{v} \mapsto \mathbf{vA}$, as usual), let M be the image of $\mathbf{I_m} - \mathbf{Q}$, and consider its preimage first by $\mathbf{P}$ and then by ρ, see the following diagram:

$$\begin{tikzcd}
\mathrm{Fix}\,\phi \leqslant F_n \arrow[r, two heads, "\rho"] & \mathbb{Z}^n \arrow[r, "\mathbf{P}"] & \mathbb{Z}^m \arrow[loop above, "\mathbf{I_m}-\mathbf{Q}"] \\
M\mathbf{P}^{-1}\rho^{-1} \arrow[u, phantom, sloped, "\trianglelefteq"] & M\mathbf{P}^{-1} \arrow[l, mapsto] \arrow[u, phantom, sloped, "\trianglelefteq"] & M = \mathrm{Im}(\mathbf{I_m} - \mathbf{Q}). \arrow[l, mapsto] \arrow[u, phantom, sloped, "\trianglelefteq"]
\end{tikzcd}$$

Equation (6.3) can be rewritten as

$$(\mathrm{Fix}\,\Psi)\pi = \mathrm{Fix}\,\phi \cap M\mathbf{P}^{-1}\rho^{-1}. \tag{6.4}$$

However, this description does not show whether $\mathrm{Fix}\,\Psi$ is finitely generated because $\mathrm{Fix}\,\phi$ is in fact finitely generated, but $M\mathbf{P}^{-1}\rho^{-1}$ is not in general. We shall avoid the intersection with $\mathrm{Fix}\,\phi$ by reducing M to a certain subgroup. Let ρ' be the restriction of ρ to $\mathrm{Fix}\,\phi$ (not to be confused with the abelianization map of the subgroup $\mathrm{Fix}\,\phi$ itself), let $\mathbf{P'}$ be the restriction of $\mathbf{P}$ to $\mathrm{Im}\,\rho'$, and let $N = M \cap \mathrm{Im}\,\mathbf{P'}$, see the following diagram:

$$\begin{tikzcd}
F_n \arrow[r, two heads, "\rho"] & \mathbb{Z}^n \arrow[r, "\mathbf{P}"] & \mathbb{Z}^m \arrow[loop above, "\mathbf{I_m}-\mathbf{Q}"] \geqslant M = \mathrm{Im}(\mathbf{I_m} - \mathbf{Q}) \\
\mathrm{Fix}\,\phi \arrow[u, phantom, sloped, "\leqslant"] \arrow[r, two heads, "\rho'"] & \mathrm{Im}\,\rho' \arrow[u, phantom, sloped, "\trianglelefteq"] \arrow[r, two heads, "\mathbf{P'}"] & \mathrm{Im}\,\mathbf{P'} \arrow[u, phantom, sloped, "\trianglelefteq"] \\
N\mathbf{P'}^{-1}\rho'^{-1} \arrow[u, phantom, sloped, "\trianglelefteq"] \arrow[d, phantom, sloped, "="] & N\mathbf{P'}^{-1} \arrow[l, mapsto] \arrow[u, phantom, sloped, "\trianglelefteq"] & N = M \cap \mathrm{Im}\,\mathbf{P'}. \arrow[l, mapsto] \arrow[u, phantom, sloped, "\trianglelefteq"] \\
(\mathrm{Fix}\,\Psi)\pi & &
\end{tikzcd}$$

Equation (6.4) then rewrites into

$$(\mathrm{Fix}\,\Psi)\pi = N\mathbf{P'}^{-1}\rho'^{-1}.$$

Now, since $N\mathbf{P'}^{-1}\rho'^{-1}$ is a normal subgroup of $\mathrm{Fix}\,\phi$ (not, in general, of F_n), it is finitely generated if and only if it is either trivial, or of finite index in $\mathrm{Fix}\,\phi$.

Note that ρ' is injective (and thus bijective) if and only if $\mathrm{Fix}\,\phi$ is either trivial, or cyclic not abelianizing to zero (indeed, for this to be the case we cannot have two freely independent elements in $\mathrm{Fix}\,\phi$ and so, $\mathrm{rk}(\mathrm{Fix}\,\phi) \leqslant 1$). Thus,

$(\operatorname{Fix}\Psi)\pi = N\mathbf{P'}^{-1}\rho'^{-1} = 1$ if and only if $\operatorname{Fix}\phi$ is trivial or cyclic not abelianizing to zero, and $N\mathbf{P'}^{-1} = \{\mathbf{0}\}$.

On the other side, by Lemma 3.2 (ii), $N\mathbf{P'}^{-1}\rho'^{-1}$ has finite index in $\operatorname{Fix}\phi$ if and only if N has finite index in $\operatorname{Im}\mathbf{P'}$ i.e. if and only if $\operatorname{rk}(N) = \operatorname{rk}(\operatorname{Im}\mathbf{P'})$. □

Example 6.5. Let us analyze again the example given at the beginning of this section, under the light of the Theorem 6.4. We considered the automorphism Ψ of $\mathbb{Z} \times F_2 = \langle t \mid \rangle \times \langle a, b \mid \rangle$ given by $a \mapsto ta$, $b \mapsto b$ and $t \mapsto t$, i.e. $\Psi = \Psi_{I_2,\mathbf{I_1},\mathbf{P}}$, where $\mathbf{P}$ is the 2×1 matrix $\mathbf{P} = (\mathbf{1},\mathbf{0})^{\top}$. It is clear that $\operatorname{Fix}(I_2) = F_2$ and so, conditions (i) and (ii) from Proposition 6.4 do not hold. Furthermore, $\rho' = \rho$, $\mathbf{P'} = \mathbf{P}$, $M = \operatorname{Im}(\mathbf{0}) = \{\mathbf{0}\}$, $N = \{\mathbf{0}\}$, while $\operatorname{Im}\mathbf{P'} = \mathbb{Z}$; hence, condition (iii) from Theorem 6.4 does not hold either, according to the fact that $\operatorname{Fix}\Psi$ is not finitely generated.

Finally, the proof of Theorem 6.4 is explicit enough to allow us to make the whole thing algorithmic: given a type (I) endomorphism $\Psi = \Psi_{\phi,\mathbf{Q},\mathbf{P}} \in \operatorname{End}(\mathbb{Z}^m \times F_n)$, the decision on whether $\operatorname{Fix}\Psi$ if finitely generated or not, and the computation of a basis for it in case it is, can be made effective assuming we have a procedure to compute a (free) basis for $\operatorname{Fix}\phi$:

Proposition 6.6. *Let $G = \mathbb{Z}^m \times F_n$ with $n \neq 1$, and let $\Psi = \Psi_{\phi,\mathbf{Q},\mathbf{P}}$ be a type (I) endomorphism of G. Assuming a (finite and free) basis for $\operatorname{Fix}\phi$ is given to us, we can algorithmically decide whether $\operatorname{Fix}\Psi$ is finitely generated or not and, in case it is, compute a basis for it.*

Proof. Let $\{v_1, \ldots, v_p\}$ be the (finite and free) basis for $\operatorname{Fix}\phi \leqslant F_n$ given to us in the hypothesis.

Theorem 6.4 describes how is $\operatorname{Fix}\Psi$ and when is it finitely generated. Assuming the notation from the proof there, we can compute abelian bases for $N \leqslant \operatorname{Im}\mathbf{P'} \leqslant \mathbb{Z}^m$ and $N\mathbf{P'}^{-1} \leqslant \operatorname{Im}\rho' \leqslant \mathbb{Z}^n$. Then, we can easily check whether any of the following three conditions hold:

(i) $\operatorname{Fix}\phi$ is trivial,

(ii) $\operatorname{Fix}\phi = \langle z \rangle$, $z\rho \neq \mathbf{0}$ and $N\mathbf{P'}^{-1} = \{\mathbf{0}\}$,

(iii) $\operatorname{rk}(N) = \operatorname{rk}(\operatorname{Im}\mathbf{P'})$.

If (i), (ii) and (iii) fail then $\operatorname{Fix}\Psi$ is not finitely generated and we are done. Otherwise, $\operatorname{Fix}\Psi$ is finitely generated and it remains to compute a basis. From (1.5), we have

$$\operatorname{Fix}\Psi = \big((\operatorname{Fix}\Psi) \cap \mathbb{Z}^m\big) \times (\operatorname{Fix}\Psi)\pi\alpha,$$

where $\operatorname{Fix}\Psi \xleftarrow{\alpha} (\operatorname{Fix}\Psi)\pi$ is any splitting of $\pi_{|\operatorname{Fix}\Psi}\colon \operatorname{Fix}\Psi \twoheadrightarrow (\operatorname{Fix}\Psi)\pi$. We just have to compute a basis for each part and put them together (after algorithmically computing some splitting α). Regarding the abelian part, equation (6.2) tells us that

$$(\operatorname{Fix}\Psi) \cap \mathbb{Z}^m = \{\mathbf{t^a} \mid \mathbf{a}(\mathbf{I_m} - \mathbf{Q}) = \mathbf{0}\},$$

and we can easily find an abelian basis for it by just computing $\ker(\mathbf{I_m} - \mathbf{Q})$.

Consider now the free part. In cases (i) and (ii), $(\operatorname{Fix}\Psi)\pi = 1$ and there is nothing to compute. Note that, in these cases, $\operatorname{Fix}\Psi$ is then an abelian subgroup of $\mathbb{Z}^m \times F_n$.

Assume case (iii), i.e. $\operatorname{rk}(N) = \operatorname{rk}(\operatorname{Im}\mathbf{P'})$. In this situation, N has finite index in $\operatorname{Im}\mathbf{P'}$ and so, $N\mathbf{P'}^{-1}$ has finite index in $\operatorname{Im}\rho'$; let us compute a set of coset representatives of $\operatorname{Im}\rho'$ modulo $N\mathbf{P'}^{-1}$,

$$\operatorname{Im}\rho' = (N\mathbf{P'}^{-1})\mathbf{c_1} \sqcup \cdots \sqcup (N\mathbf{P'}^{-1})\mathbf{c_q},$$

(see Section 3). Now, according to Lemma 3.2 (b), we can transfer this partition via ρ' to obtain a system of right coset representatives of $\operatorname{Fix}\phi$ modulo $(\operatorname{Fix}\Psi)\pi = N\mathbf{P'}^{-1}\rho'^{-1}$,

$$\operatorname{Fix}\phi = (N\mathbf{P'}^{-1}\rho'^{-1})z_1 \sqcup \cdots \sqcup (N\mathbf{P'}^{-1}\rho'^{-1})z_q. \tag{6.5}$$

To compute the z_i's, note that $\mathbf{v_1} = v_1\rho', \ldots, \mathbf{v_p} = v_p\rho'$ generate $\operatorname{Im}\rho'$, write each $\mathbf{c_i} \in \operatorname{Im}\rho'$ as a (non necessarily unique) linear combination of them, say $\mathbf{c_i} = c_{i,1}\mathbf{v_1} + \cdots + c_{i,p}\mathbf{v_p}$, $i \in [q]$, and take $z_i = v_1^{c_{i,1}} v_2^{c_{i,1}} \cdots v_p^{c_{i,p}} \in \operatorname{Fix}\phi$.

Now, construct a free basis for $N\mathbf{P'}^{-1}\rho'^{-1} = (\operatorname{Fix}\Psi)\pi$ following the first of the two alternatives at the end of the proof of Theorem 4.8 (the second one does not work here because ρ' is not the abelianization of the subgroup $\operatorname{Fix}\phi$, but the restriction there of the abelianization of F_n):

Build the Schreier graph $\mathcal{S}(N\mathbf{P'}^{-1}\rho'^{-1})$ for $N\mathbf{P'}^{-1}\rho'^{-1} \leqslant \operatorname{Fix}\phi$ with respect to $\{v_1, \ldots, v_p\}$, in the following way: consider the graph with the cosets of (6.5) as vertices, and with no edge. Then, for every vertex $(N\mathbf{P'}^{-1}\rho'^{-1})z_i$ and every letter v_j, add an edge labeled v_j from $(N\mathbf{P'}^{-1}\rho'^{-1})z_i$ to $(N\mathbf{P'}^{-1}\rho'^{-1})z_i v_j$, algorithmically identified among the available vertices by repeatedly using the membership problem for $N\mathbf{P'}^{-1}\rho'^{-1}$ (note that we can easily do this by abelianizing the candidate and checking whether it belongs to $N\mathbf{P'}^{-1}$). Once we have run over all i, j, we shall get the full graph $\mathcal{S}(N\mathbf{P'}^{-1}\rho'^{-1})$, from which we can easily obtain a free basis for $N\mathbf{P'}^{-1}\rho'^{-1} = (\operatorname{Fix}\Psi)\pi$.

Finally, having a free basis for $(\operatorname{Fix}\Psi)\pi$, we can easily construct an splitting $\operatorname{Fix}\Psi \xleftarrow{\alpha} (\operatorname{Fix}\Psi)\pi$ for $\pi_{|\operatorname{Fix}\Psi}\colon \operatorname{Fix}\Psi \twoheadrightarrow (\operatorname{Fix}\Psi)\pi$ by just computing, for each generator $u \in (\operatorname{Fix}\Psi)\pi$, a preimage $\mathbf{t^a}u \in \operatorname{Fix}\Psi$, where $\mathbf{a} \in \mathbb{Z}^m$ is a completion found by solving the system of equations $\mathbf{a}(\mathbf{I_m} - \mathbf{Q}) = \mathbf{uP}$ (see (6.2)).

This completes the proof. □

Bringing together Propositions 6.3 and 6.6 and Theorem 6.4, we get the following.

Corollary 6.7. *For $m \geqslant 1$ and $n \geqslant 2$,*

(i) *if $\mathrm{FPP}_{\mathbf{a}}(F_n)$ is solvable then $\mathrm{FPP}_{\mathbf{a}}(\mathbb{Z}^m \times F_n)$ is also solvable.*

(ii) *if $\mathrm{FPP}_{\mathbf{e}}(F_n)$ is solvable then $\mathrm{FPP}_{\mathbf{e}}(\mathbb{Z}^m \times F_n)$ is also solvable.* □

To close this section, we point the reader to some very recent results related to fixed subgroups of endomorphisms of partially commutative groups. In [22], E. Rodaro, P.V. Silva and M. Sykiotis characterize which partially commutative groups G satisfy that $\operatorname{Fix}\Psi$ is finitely generated for every $\Psi \in \operatorname{End}(G)$ (and, of course, free-abelian times free groups are not included there); they also provide similar results concerning automorphisms.

7 The Whitehead problem

J. Whitehead, back in the 30's of the last century, gave an algorithm [27] to decide, given two elements u and v from a finitely generated free group F_n, whether there exists an automorphism $\phi \in \operatorname{Aut}(F_n)$ sending one to the other, $v = u\phi$. Whitehead's algorithm uses a (today) very classical piece of combinatorial group theory technique called 'peak reduction', see also [18]. Several variations of this problem (like replacing u and v by tuples of words, relaxing equality to equality up to conjugacy, adding conditions on the conjugators, replacing words by subgroups, replacing automorphisms to monomorphisms or endomorphisms, etc), as well as extensions of all these problems to other families of groups, can be found in the literature, all of them generally known as the *Whitehead problem*. Let us consider here the following ones for an arbitrary finitely generated group G:

Problem 7.1 (**Whitehead Problem,** $\mathrm{WhP}_{\mathbf{a}}(G)$)**.** *Given two elements $u, v \in G$, decide whether there exist an automorphism ϕ of G such that $u\phi = v$, and, if so, find one (giving the images of the generators).*

Problem 7.2 (**Whitehead Problem,** $\mathrm{WhP}_{\mathbf{m}}(G)$)**.** *Given two elements $u, v \in G$, decide whether there exist a monomorphism ϕ of G such that $u\phi = v$, and, if so, find one (giving the images of the generators).*

Problem 7.3 (**Whitehead Problem,** $\mathrm{WhP}_{\mathbf{e}}(G)$)**.** *Given two elements $u, v \in G$, decide whether there exist an endomorphism ϕ of G such that $u\phi = v$, and, if so, find one (giving the images of the generators).*

In this last section we shall solve these three problems for free-abelian times free groups. We note that, very recently, a new version of the classical peak-reduction theorem has been developed by M. Day [10] for an arbitrary partially commutative group, see also [9]. These techniques allow the author to solve the Whitehead problem for partially commutative groups, in its variant relative to automorphisms and tuples of conjugacy classes. In particular $\mathrm{WhP}_{\mathbf{a}}(G)$ (which was conjectured in [9]) is solved in [10] for any partially commutative group G. As far as we know, $\mathrm{WhP}_{\mathbf{m}}(G)$ and $\mathrm{WhP}_{\mathbf{e}}(G)$ remain unsolved in general. Our Theorem 7.6 below is a small contribution into this direction, solving these problems for free-abelian times free groups.

Let us begin by reminding the situation of the Whitehead problems for free-abelian and for free groups (the first one is folklore, and the second one is well-known). The following lemma is straightforward to prove and, in particular, solves

$\mathrm{WhP}_{\mathbf{a}}(\mathbb{Z}^m)$, $\mathrm{WhP}_{\mathbf{m}}(\mathbb{Z}^m)$ and $\mathrm{WhP}_{\mathbf{e}}(\mathbb{Z}^m)$. Here, for a vector $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, we write $\gcd(\mathbf{a})$ to denote the greatest common divisor of the a_i's (with the convention that $\gcd(\mathbf{0}) = 0$).

Lemma 7.4. *If* $\mathbf{u} \in \mathbb{Z}^n$ *and* $\mathbf{a} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$, *then*

(i) $\{\mathbf{aQ} \mid \mathbf{Q} \in \mathrm{GL}_m(\mathbb{Z})\} = \{\mathbf{a}' \in \mathbb{Z}^m \mid \gcd(\mathbf{a}) = \gcd(\mathbf{a}')\}$,

(ii) $\{\mathbf{aQ} \mid \mathbf{Q} \in \mathcal{M}_m(\mathbb{Z})$ *with* $\det(\mathbf{Q}) \neq 0\} = \{\mathbf{a}' \in \mathbb{Z}^m \mid \gcd(\mathbf{a}) \mid \gcd(\mathbf{a}')\} \setminus \{\mathbf{0}\}$,

(iii) $\{\mathbf{uP} \mid \mathbf{P} \in \mathcal{M}_{n\times m}(\mathbb{Z})\} = \{\mathbf{u}' \in \mathbb{Z}^m \mid \gcd(\mathbf{u}) \mid \gcd(\mathbf{u}')\}$. □

As expected, the same problems for the free group F_n are much more complicated. As mentioned above, the case of automorphisms was already solved by Whitehead back in the 30's of last century. The case of endomorphisms can be solved by writing a system of equations over F_n (with unknowns being the images of a given free basis for F_n), and then solving it by the powerful Makanin's algorithm. Finally, the case of monomorphisms was recently solved by Ciobanu-Houcine.

Theorem 7.5. *For* $n \geqslant 2$,

(i) [**Whitehead,** [27]] $\mathrm{WhP}_{\mathbf{a}}(F_n)$ *is solvable.*

(ii) [**Ciobanu-Houcine,** [8]] $\mathrm{WhP}_{\mathbf{m}}(F_n)$ *is solvable.*

(iii) [**Makanin,** [19]] $\mathrm{WhP}_{\mathbf{e}}(F_n)$ *is solvable.* □

Theorem 7.6. *Let* $m \geqslant 1$ *and* $n \geqslant 2$, *then*

(i) $\mathrm{WhP}_{\mathbf{a}}(\mathbb{Z}^m \times F_n)$ *is solvable.*

(ii) $\mathrm{WhP}_{\mathbf{m}}(\mathbb{Z}^m \times F_n)$ *is solvable.*

(iii) $\mathrm{WhP}_{\mathbf{e}}(\mathbb{Z}^m \times F_n)$ *is solvable.*

Proof. We are given two elements $\mathbf{t}^{\mathbf{a}}u, \mathbf{t}^{\mathbf{b}}v \in G = \mathbb{Z}^m \times F_n$, and have to decide whether there exists an automorphism (resp. monomorphism, endomorphism) of G sending one to the other. And in the affirmative case, find one of them. For convenience, we shall prove (ii), (i) and (iii) in this order.

(ii). Since all monomorphisms of G are of type (I), we have to decide whether there exist a monomorphism ϕ of F_n, and matrices $\mathbf{Q} \in \mathcal{M}_m(\mathbb{Z})$ and $\mathbf{P} \in \mathcal{M}_{n\times m}(\mathbb{Z})$, with $\det \mathbf{Q} \neq 0$, such that $(\mathbf{t}^{\mathbf{a}}u)\Psi_{\phi,\mathbf{Q},\mathbf{P}} = \mathbf{t}^{\mathbf{b}}v$. Separating the free and free-abelian parts, we get two independent problems:

$$\left.\begin{aligned} u\phi &= v \\ \mathbf{aQ} + \mathbf{uP} &= \mathbf{b} \end{aligned}\right\} \tag{7.1}$$

On one hand, we can use Theorem 7.5 (ii) to decide whether there exists a monomorphism ϕ of F_n such that $u\phi = v$. If not then our problem has no solution either, and we are done; otherwise, $\mathrm{WhP}_{\mathbf{m}}(F_n)$ gives us such a ϕ.

On the other hand, we need to know whether there exist matrices $\mathbf{Q} \in \mathcal{M}_m(\mathbb{Z})$ and $\mathbf{P} \in \mathcal{M}_{n\times m}(\mathbb{Z})$, with $\det \mathbf{Q} \neq 0$ and such that $\mathbf{aQ} + \mathbf{uP} = \mathbf{b}$, where $\mathbf{u} \in \mathbb{Z}^n$ is the abelianization of $u \in F_n$ (given from the beginning). If $\mathbf{a} = \mathbf{0}$ or $\mathbf{u} = \mathbf{0}$, this is already solved in Lemma 7.4(iii) or (ii). Otherwise, write $0 \neq \alpha = \gcd(\mathbf{a})$ and $0 \neq \mu = \gcd(\mathbf{u})$; and, according to Lemma 7.4, we have to decide whether there exist $\mathbf{a}' \in \mathbb{Z}^m$ and $\mathbf{u}' \in \mathbb{Z}^m$, with $\mathbf{a}' \neq \mathbf{0}$, $\alpha \mid \gcd(\mathbf{a}')$, and $\mu \mid \gcd(\mathbf{u}')$, such that $\mathbf{a}' + \mathbf{u}' = \mathbf{b}$. Writing $\mathbf{a}' = \alpha\,\mathbf{x}$ and $\mathbf{u}' = \mu\,\mathbf{y}$, the problem reduces to test whether the following linear system of equations

$$\left.\begin{array}{ccccc} \alpha\,x_1 & + & \mu\,y_1 & = & b_1 \\ & \vdots & & \vdots & \\ \alpha\,x_m & + & \mu\,y_m & = & b_m \end{array}\right\} \tag{7.2}$$

has any integral solution $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathbb{Z}$ such that $(x_1, \ldots, x_m) \neq \mathbf{0}$. A necessary and sufficient condition for the system (7.2) to have a solution is $\gcd(\alpha, \mu) \mid b_j$, for every $j \in [m]$. And note that, if (x_1, y_1) is a solution to the first equation, then $(x_1 + \mu, y_1 - \alpha)$ is another one; since $\mu \neq 0$, the condition $(x_1, \ldots, x_m) \neq \mathbf{0}$ is then superfluous. Therefore, the answer is affirmative if and only if $\gcd(\alpha, \mu) \mid b_j$, for every $j \in [m]$; and, in this case, we can easily reconstruct a monomorphism Ψ of G such that $(\mathbf{t}^{\mathbf{a}} u)\Psi = \mathbf{t}^{\mathbf{b}} v$.

(i). The argument for automorphisms is completely parallel to the previous discussion replacing the conditions ϕ monomorphism and $\det \mathbf{Q} \neq 0$, to ϕ automorphism and $\det \mathbf{Q} = \pm 1$. We manage the first change by using Theorem 7.5 (i) instead of (ii). The second change forces us to look for solutions to the linear system (7.2) with the extra requirement $\gcd(\mathbf{x}) = 1$ (because now $\gcd(\mathbf{a}')$ should be equal and not just multiple of α).

So, if any of the conditions $\gcd(\alpha, \mu) \mid b_j$ fails, the answer is negative and we are done. Otherwise, write $\rho = \gcd(\alpha, \mu)$, $\alpha = \rho\alpha'$ and $\mu = \rho\mu'$, and the general solution for the j-th equation in (7.2) is

$$(x_j, y_j) = (x_j^0, y_j^0) + \lambda_j(\mu', -\alpha'), \quad \lambda_j \in \mathbb{Z},$$

where (x_j^0, y_j^0) is a particular solution, which can be easily computed. Thus, it only remains to decide whether there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}$ such that

$$\gcd(x_1^0 + \lambda_1\mu', \, \ldots, \, x_m^0 + \lambda_m\mu') = 1. \tag{7.3}$$

We claim that this happens if and only if

$$\gcd(x_1^0, \, \ldots, \, x_m^0, \, \mu') = 1, \tag{7.4}$$

which is clearly a decidable condition.

Reorganizing a Bezout identity for (7.3) we can obtain a Bezout identity for (7.4). Hence (7.3) implies (7.4). For the converse, assume the integers $x_1^0, \ldots, x_m^0, \mu'$ are coprime, and we can fulfill equation (7.3) by taking $\lambda_1 = \cdots = \lambda_{m-1} = 0$ and λ_m equal to the product of the primes dividing $x_1^0, \ldots, x_{m-1}^0$ but not x_m^0 (take $\lambda_m = 1$ if there is no such prime). Indeed, let us see that any prime

p dividing $x_1^0, \dots, x_{m-1}^0$ is not a divisor of $x_m^0 + \lambda_m \mu'$. If p divides x_m^0, then p does not divide neither μ' nor λ_m and therefore $x_m^0 + \lambda_m \mu'$ either. If p does not divide x_m^0, then p divides λ_m by construction, hence p does not divide $x_m^0 + \lambda_m \mu'$. This completes the proof of the claim, and of the theorem for automorphisms.

(iii). In our discussion now, we should take into account endomorphisms of both types.

Again, the argument to decide whether there exists an endomorphism of type (I) sending $\mathbf{t}^{\mathbf{a}} u$ to $\mathbf{t}^{\mathbf{b}} v$, is completely parallel to the above proof of (ii), replacing the condition ϕ monomorphism to ϕ endomorphism, and deleting the condition $\det \mathbf{Q} \neq 0$ (and allowing here an arbitrary matrix $\mathbf{Q}$). We manage the first change by using Theorem 7.5 (iii) instead of (ii). The second change simply leads us to solve the system (7.2) with no extra condition on the variables; so, the answer is affirmative if and only if $\gcd(\alpha, \mu) \mid b_j$, for every $j \in [m]$.

It remains to consider endomorphisms of type (II), $\Psi_{z,\mathbf{l},\mathbf{h},\mathbf{Q},\mathbf{P}}$. So, given our elements $\mathbf{t}^{\mathbf{a}} u$ and $\mathbf{t}^{\mathbf{b}} v$, and separating the free and free-abelian parts, we have to decide whether there exist $z \in F_n$, $\mathbf{l} \in \mathbb{Z}^m$, $\mathbf{h} \in \mathbb{Z}^n$, $\mathbf{Q} \in \mathcal{M}_m(\mathbb{Z})$, and $\mathbf{P} \in \mathcal{M}_{n \times m}(\mathbb{Z})$ such that

$$\left.\begin{aligned} z^{\mathbf{a}\mathbf{l}^\top + \mathbf{u}\mathbf{h}^\top} &= v \\ \mathbf{a}\mathbf{Q} + \mathbf{u}\mathbf{P} &= \mathbf{b} \end{aligned}\right\} \tag{7.5}$$

(note that we can ignore the condition $\mathbf{l} \neq \mathbf{0}$ because if $\mathbf{l} = \mathbf{0}$ then the endomorphism becomes of type (I) as well, and this case is already considered before). Again the two equations are independent. About the free part, note that the integers $\mathbf{a}\mathbf{l}^\top + \mathbf{u}\mathbf{h}^\top$ with $\mathbf{l} \in \mathbb{Z}^m$ and $\mathbf{h} \in \mathbb{Z}^n$ are precisely the multiples of $d = \gcd(\mathbf{a}, \mathbf{u})$; so, it has a solution if and only if v is a d^{th} power in F_n, a very easy condition to check. And about the second equation, it is exactly the same as when considering endomorphisms of type (I), so its solvability is already discussed. □

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